Moving the quantifiers to the front! (Prenex Form.)

We have discussed how to decide the truth of a statement with the quantifiers in the front. But what if they are not in the front?

$$((\exists y)(\forall x)(x = y)) \to ((\forall x)(\exists y)(x = y))$$

We have rules to move quantifiers to the front, without altering the meaning.

(1) $\neg(\forall x)P \;\equiv\; (\exists x)(\neg P)$.
(2) $\neg(\exists x)P \;\equiv\; (\forall x)(\neg P)$.
(3) $P \vee ((\exists x)Q) \;\equiv\; (\exists x)(P \vee Q)$ if P does not depend on x.
(4) $P \vee ((\forall x)Q) \;\equiv\; (\forall x)(P \vee Q)$ if P does not depend on x.
(5) $P \wedge ((\exists x)Q) \;\equiv\; (\exists x)(P \wedge Q)$ if P does not depend on x.
(6) $P \wedge ((\forall x)Q) \;\equiv\; (\forall x)(P \wedge Q)$ if P does not depend on x.

For example,

$$((\exists y)(\forall x)(x = y)) \to ((\forall x)(\exists y)(x = y)) \;\equiv\; (\forall s)(\exists t)(\forall x)(\exists y)((s = t) \to (x = y))$$

Practice!

Write the following statements in a logically equivalent form with quantifiers at the front.

(1) If every lumberjack is hungry, then some lumberjack is hungry.

(2) $(\exists x)P(x) \leftrightarrow (\exists x)Q(x)$

(3) $(\forall x)(\forall y)((x < y) \to (\exists z)(x < z < y))$

Restricted quantifiers!

Often one sees something like

$$(\forall \varepsilon > 0)(\exists \delta > 0)(|x - a| < \varepsilon \to |f(x) - f(a)| < \delta).$$

Question: What does it mean to write $(\forall \varepsilon > 0)$? More generally, if $C(x)$ is a condition on x and $P(x)$ is a statement about x, what does $((\forall x)C(x))P(x)$ mean?

Answer: $((\forall x)C(x))P(x)$ is an abbreviation for

$$(\forall x)(C(x) \to P(x)),$$

and $((\exists x)C(x))P(x)$ is an abbreviation for

$$(\exists x)(C(x) \land P(x)).$$

We call $((\forall x)C(x))$ and $((\exists x)C(x))$ *restricted quantifiers.* They behave just like ordinary quantifiers in the sense that

(1) Rules for logical equivalence are the same:
   (a) $\neg((\forall x)C(x))P(x) \equiv ((\exists x)C(x))(\neg P(x))$.
   (b) $P \lor (((\exists x)C(x))Q(x)) \equiv ((\exists x)C(x))(P \lor Q(x))$ if P does not depend on x.
   (c) ETC
(2) Quantifier games are played the same way. For example, to determine the truth of

$$(\forall x > 0)(\exists y < 1)(x < y)$$

we play a game where $\forall$ first chooses x *satisfying the condition* $x > 0$, then $\exists$ chooses y *satisfying the condition* $y < 1$.

Practice!

(1) Is $(\forall x > 0)(\exists y < 1)(x < y)$ true in $\mathbb{R}$? In $\mathbb{N}$? In each case give a strategy for the appropriate quantifier.

(2) Move the restricted quantifiers to the front: $(\exists x > 0)P(x) \leftrightarrow (\exists x > 0)Q(x)$