Practice with tables! (Solutions in blue!)

Let $\mathbb{A} = \langle U, V; +, \diamond, \Box, \sqsubseteq \rangle$ be a structure where

(1) $U = \{a, b\}$, $V = \{p, q\}$,
(2) $+ : U \times V \to U$ is a binary operation from U and V to U,
(3) $\diamond : U \times U \to V$ is a binary operation from U to V,
(4) $\Box : U \to V$ is a unary operation from U to V,
(5) $\sqsubseteq : V \times V \to \{\top, \bot\}$ is a binary predicate.

Suppose the tables for these structural elements are

x	y	$x + y$
a	p	a
a	q	a
b	p	b
b	q	a

x	y	$x \diamond y$
a	a	p
a	b	q
b	a	q
b	b	q

x	$\Box x$
a	q
b	p

x	y	$x \sqsubseteq y$
p	p	$\top$
p	q	$\bot$
q	p	$\bot$
q	q	$\top$

Create tables for these compound structural elements. If you have time, draw tree representations.

(1) The compound operation $(x \diamond (x + \Box x))$.

x	$\Box x$	$x + \Box x$	$x \diamond (x + \Box x)$
a	q	a	p
b	p	b	q

(2) The compound operation $((x + y) \diamond (x + z))$.

Since x is a left input for $+$, we must have $x \in \{a, b\}$. Since y and z are right inputs for $+$, we must have $y, z \in \{p, q\}$.

x	y	z	$x + y$	$x + z$	$(x + y) \diamond (x + z)$
a	p	p	a	a	p
a	p	q	a	a	p
a	q	p	a	a	p
a	q	q	a	a	p
b	p	p	b	b	q
b	p	q	b	a	q
b	q	p	a	b	q
b	q	q	a	a	p

(3) The compound predicate $\Box x \sqsubseteq (x \diamond x)$.
(This could be written in prefix notation as $\sqsubseteq(\Box(x), \diamond(x, x))$.)

x	$\Box x$	$(x \diamond x)$	$\Box x \sqsubseteq (x \diamond x)$
a	q	p	$\bot$
b	p	q	$\bot$