Ordered Pairs, Relations

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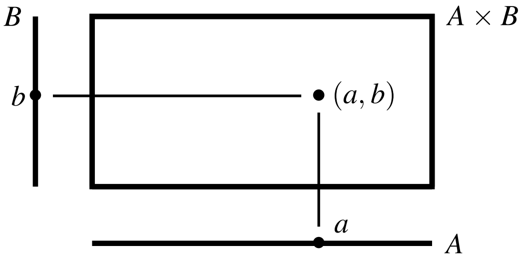

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is a binary relation on Z that records the pairs of siblings. Thus, binary relations can be used to record the information of concepts depending on two things.

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Paul	$*$	$*$	$*$	$*$
George	$*$	$*$	$*$	
Ringo			$*$	$*$

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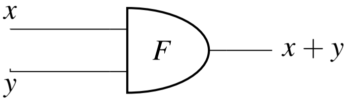

$b = x + y$

A visual represention a function

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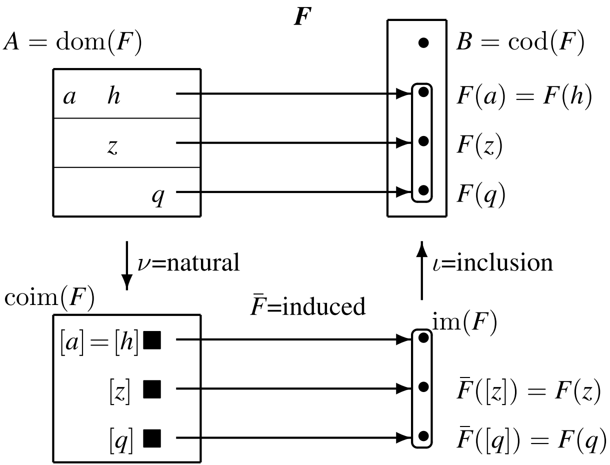

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2. The **preimage** or **inverse image** of a subset $V \subseteq B$ is $F^{-1}[V] = \{a \in A \mid F(a) \in V\}$.
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8. The **canonical factorization of** F is $F = \iota \circ \bar{F} \circ \nu$.