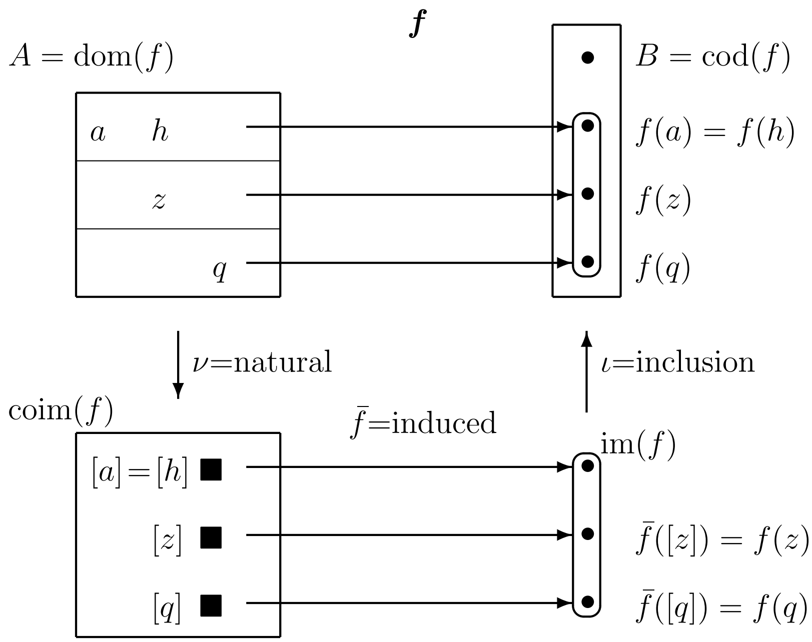
Notes about functions.

Let A and B be sets and let $f\colon A \to B$ be a function from A to B. There are sets and functions related to A, B and f that have special names.



(1) The *image* of f is $\mathrm{im}(f) = f[A] = \{b \in B \ : \ \exists a \in A(f(a) = b)\}$. The image of a subset $U \subseteq A$ is $f[U] = \{b \in B \ : \ \exists u \in U(f(u) = b)\}$.
(2) The *preimage* or *inverse image* of a subset $V \subseteq B$ is $f^{-1}[V] = \{a \in A \ : \ f(a) \in V\}$.
(3) The preimage of a singleton $\{b\}$ is written $f^{-1}(b)$ and sometimes called the *fiber* of f *over* b. The fiber containing the element a is sometimes written $[a]$.
(4) The *coimage* of f is the set $\mathrm{coim}(f) = \{f^{-1}(b) \ : \ b \in \mathrm{im}(f)\}$ of all fibers.
(5) The *natural map* is $\nu\colon A \to \mathrm{coim}(f)\colon a \mapsto [a]$. (This says $\nu(a) = [a]$.)
(6) The *inclusion map* is $\iota\colon \mathrm{im}(f) \to B\colon b \mapsto b$. (This says $\iota(b) = b$.)
(7) The *induced map* is $\bar{f}\colon \mathrm{coim}(f) \to \mathrm{im}(f)\colon [a] \mapsto f(a)$. (This says $\bar{f}([a]) = f(a)$.)

Some facts:

(1) The natural map is *surjective*.
(2) The inclusion map is *injective*.
(3) The induced map is *bijective*.
(4) $f = \iota \circ \bar{f} \circ \nu$. (This is the *canonical factorization* of f.)

More Terminology about Functions

(1) $F \subseteq A \times B$, $\quad F\colon A \to B$, $\quad A \xrightarrow{F} B$.

The first notation expresses only that F is a binary relation from A to B. The second and third notation express that F is a function from A to B, so it is a binary realtion from A to B that satisfies the function rule.

(2) F assigns y to x, $\quad y = F(x)$.

This is to remind us that if $F(x) = y$, then F is assigning to x the value y, not the other way around. (F does not assign x to y, rather it assigns y to x.)

(3) $F\colon A \to B\colon x \mapsto$ (value assigned to x). (E.g., $F\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto x^2$)

This is a description of the "mapsto" symbol, $\mapsto$. This is not simply another type of arrow that can be used interchangeably with $\to$. Rather, the notation

$$F\colon \mathbb{R} \to [-1, 1]\colon x \mapsto \sin(x)$$

is expressing that F is a function from the domain $\mathbb{R}$ to the codomain $[-1, 1]$ which assigns the value $\sin(x)$ to x. The $\mapsto$ symbol is used to indicate the "formula" or "rule" that defines F.

(4) F is injective: (Equivalently, F is 1-1.)

F is injective if

$$F(a) = F(b) \quad \text{implies} \quad a = b.$$

In the contrapositive (hence equivalent) form, this reads

$$a \neq b \quad \text{implies} \quad F(a) \neq F(b).$$

(5) F is surjective: (Equivalently, F is onto.)

F is surjective if $\operatorname{im}(F) = \operatorname{cod}(F)$. If we refer to the directed graph representation of F, it says that each element of the codomain "receives an arrow head". More formally, in symbols,

$$(\forall b)(\exists a)(b = F(a)).$$

Here b is a variable representing values in the codomain of F and a is a variable representing values in the domain of F.

(6) F is bijective: (Equivalently, F is 1-1 and onto.)

bijective = injective + surjective.

(7) F is the identity function on A:

The identity function on A, written id_A, is the function $\mathrm{id}_A\colon A \to A\colon x \mapsto x$. As a relation, it is

$$\mathrm{id}_A = \{(a,a) \in A^2 \mid a \in A\}.$$

(8) F is invertible:

$F\colon A \to B$ is invertible if there is a function $G\colon B \to A$ such that $G \circ F = \mathrm{id}_A$ and $F \circ G = \mathrm{id}_B$.

(9) F is constant:

$F\colon A \to B$ is constant if it assigns all elements of the domain the same value, i.e., it"assumes only one value". More precisely, F is constant if $F \subseteq A \times B$ and $F = A \times \{b\}$ for some $b \in B$. IN symbols, we indicate F is constant by writing

$$(\forall x_1)(\forall x_2)(F(x_1) = F(x_2)).$$

(10) F is the inclusion map for a subset $A \subseteq B$:

If A is a subset of B, then the inclusion map from A to B is

$$\iota\colon A \to B\colon a \mapsto a.$$

As a set, $\iota = \mathrm{id}_A$.

(11) F is the natural map for a partition P on A:

If P is a partition of A, then the natural map from A to P is

$$\nu\colon A \to P\colon a \mapsto [a].$$

This is the function that maps $a \in A$ to the cell of P containing a.

(12) $A \xrightarrow{F} B \xrightarrow{G} C$, or $G \circ F \colon A \to C$.

Here we are writing notation for the composition of F and G. The composite function $G \circ F$ is the function $(G \circ F)(a) = G(F(a))$. We read "$G \circ F$" as "G of F" (sometimes just "G circle F"). The composition is defined by the formula

$$G \circ f = \{(a, c) \in A \times C \mid (\exists b \in B)(((a, b) \in F) \wedge ((b, c) \in G))\}.$$

Example. If $F(x) = x^2$ and $G(x) = \sin(x)$, then $G \circ F(x) = G(F(x)) = \sin(x^2)$.

Practice problems.

(1) Draw a figure like the one on the first page of these notes illustrating $f \colon \mathbb{R} \to \mathbb{R} \colon x \mapsto x^2$. Identify all the "named" sets and functions.

(2) Repeat the previous exercise for the function $g \colon \mathbb{R}^2 \to \mathbb{R} \colon (x, y) \mapsto x + y$.

(3) Repeat for the *identity function* $id \colon A \to A \colon a \mapsto a$.

(4) Repeat for the *second coordinate projection* $\pi \colon X \times Y \to Y \colon (x, y) \mapsto y$.

(5) Show that
  (a) the composition of two injective functions is injective,
  (b) the composition of two surjective functions is surjective, and
  (c) the composition of two bijective functions is bijective.

(6) Show that injective functions are *left cancellable*: if f is injective, then $f \circ g = f \circ h$ implies $g = h$.

(7) Show that surjective functions are *right cancellable*: if f is surjective, then $g \circ f = h \circ f$ implies $g = h$.

(8) Show that if $f \colon A \to B$ is a function, then $f^{-1} \colon \mathcal{P}(B) \to \mathcal{P}(A)$ is also a function. Show that f is injective iff f^{-1} is surjective, and f is surjective iff f^{-1} is injective.