The Natural Numbers

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$$\varphi_{y=S(x)}(x, y) : \quad (\forall z)((z \in y) \leftrightarrow ((z \in x) \vee (z = x))).$$

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This axiom guarantees that $\mathbb{N}$ exists.

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