Naive Set Theory versus Axiomatic Set Theory

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More precisely, we will show that if we allow the construction principle of unrestricted comprehension, then some statements of the form "$x \in y$" are neither true nor false.

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But formally, we just consider x to be "a thing", like a dot on the page, and $x \in y$ asserts that x and y are related in some way. We may write this relation as $x \to y$ or $\bullet \to \bullet$ or $\overset{x}{\bullet}\!\to\!\overset{y}{\bullet}$.

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The axioms were chosen to reflect our intuition about "unordered collections of distinct objects".

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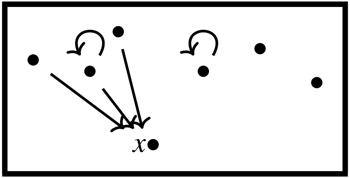

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This is just a collection of some of the vertices in D. Is C a set? For this, we would need some vertex v in D whose collection of in-neighbors is exactly C. Then v "names" the collection, and this process of "naming" certifies that C is a set.

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Question: Is R a set? That is, is there a vertex v whose set of in-neighbors is exactly R? (Key issue: If such a v exists, does it have a loop?)

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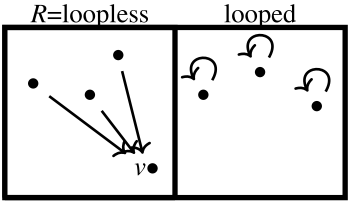

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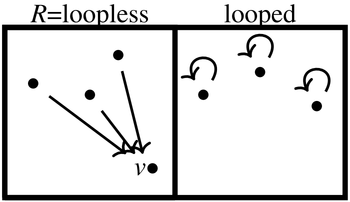

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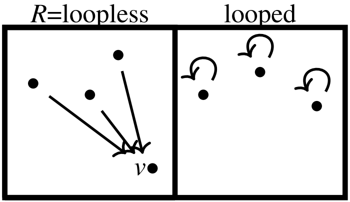

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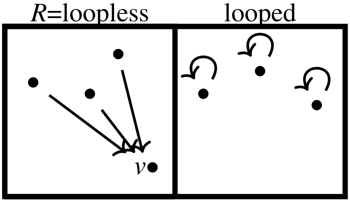

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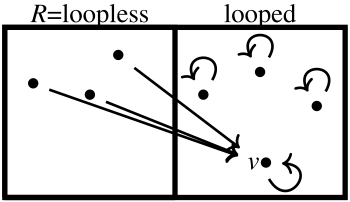

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Corollary.

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Corollary. Naive set theory is inconsistent. □

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Theorem. The class of all 1-element sets is a proper class.