$\binom{n}{k}$, $\binom{n}{k_1,\ldots,k_r}$, $\left(\!\!\binom{n}{k}\!\!\right)$

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For an alternative proof, use the formula at the top of the page.

Pascal's Triangle as an $n \times k$ table

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k

n

$\binom{n}{k}$	0	1	2	3	4	5	6	$\cdots$
0	1	0	0	0	0	0	0	
1	1	1	0	0	0	0	0	
2	1	2	1	0	0	0	0	
3	1	3	3	1	0	0	0	
4	1	4	6	4	1	0	0	
5	1	5	10	10	5	1	0	
6	1	6	16	20	16	6	1	
$\vdots$								$\ddots$

Pascal's Triangle as a triangle

Pascal's Triangle as a triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Pascal's Triangle as a triangle

$$
\begin{array}{ccccccccccccccccccccc}
&&&&&&&&&&1&&&&&&&&&& \\
&&&&&&&&&1&&1&&&&&&&&& \\
&&&&&&&&1&&2&&1&&&&&&&& \\
&&&&&&&1&&3&&3&&1&&&&&&& \\
&&&&&&1&&4&&6&&4&&1&&&&&& \\
&&&&&1&&5&&10&&10&&5&&1&&&&& \\
&&&&1&&6&&15&&20&&15&&6&&1&&&& \\
&&&1&&7&&21&&35&&35&&21&&7&&1&&& \\
&&1&&8&&28&&56&&70&&56&&28&&8&&1&& \\
&1&&9&&36&&84&&126&&126&&84&&36&&9&&1& \\
1&&10&&45&&120&&210&&252&&210&&120&&45&&10&&1
\end{array}
$$

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1 2 1
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										1										
									1		1									
								1		2		1								
							1		3		3		1							
						1		4		6		4		1						
					1		5		10		10		5		1					
				1		6		15		20		15		6		1				
			1		7		21		35		35		21		7		1			
		1		8		28		56		70		56		28		8		1		
	1		9		36		84		126		126		84		36		9		1	
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								1		2		1								
							1		3		3		1							
						1		4		6		4		1						
					1		5		10		10		5		1					
				1		6		15		20		15		6		1				
			1		7		21		35		35		21		7		1			
		1		8		28		56		70		56		28		8		1		
	1		9		36		84		126		126		84		36		9		1	
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Unimodal, since $1 \le \binom{n}{k+1} / \binom{n}{k} = \frac{k!(n-k)!}{(k+1)!(n-k-1)!} = \frac{n-k}{k+1} \Leftrightarrow k \le \frac{n-1}{2}$.

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$$\begin{aligned}(x+y)^{n+1} &= \binom{n}{0}x^{n+1}y^0 \quad + \binom{n}{1}x^n y^1 \quad + \cdots \quad + \binom{n}{n}x^1 y^n \\ &\qquad\qquad\qquad\quad + \binom{n}{0}x^n y^1 \quad + \cdots \quad + \binom{n}{n-1}x^1 y^n \quad + \binom{n}{n}x^0 y^{n+1}\end{aligned}$$

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 & & +\binom{n}{0}x^n y^1 & +\cdots & +\binom{n}{n-1}x^1y^n & +\binom{n}{n}x^0y^{n+1} \\
\hline
 & \binom{n+1}{0}x^{n+1}y^0 & +\binom{n+1}{1}x^n y^1 & +\cdots & +\binom{n+1}{n}x^1y^n & +\binom{n+1}{n+1}x^0y^{n+1}
\end{array}$$

Trinomial and multinomial coefficients

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- $\binom{n}{0,\cdots,n,\cdots,0} = 1$.

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Answer: $\binom{n}{k} \cdot \binom{n-k}{\ell} = \frac{n!}{k!(n-k)!} \cdot \frac{(n-k)!}{\ell!(n-k-\ell)!} = \frac{n!}{k!\ell!(n-k-\ell)!} = \binom{n}{k,\ell,n-k-\ell}$.

Definition. If $n = k_1 + \cdots + k_r$, then $\binom{n}{k_1,\ldots,k_r} = \frac{n!}{k_1!\cdots k_r!}$.
The combinatorial interpretation of $\binom{n}{k_1,\ldots,k_r}$ is "the number of ways to choose k_1 elements from n, then k_2 elements from the remainder, then …etc."

Recursive specification of $\binom{n}{k_1,k_2,\ldots,k_r}$.

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Pascal's Pyramid

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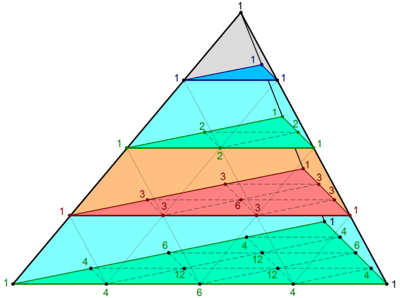
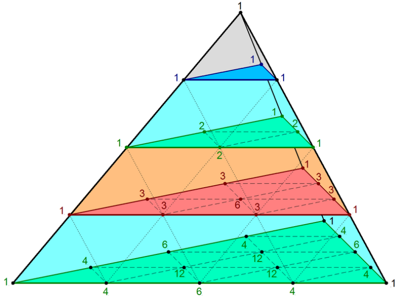
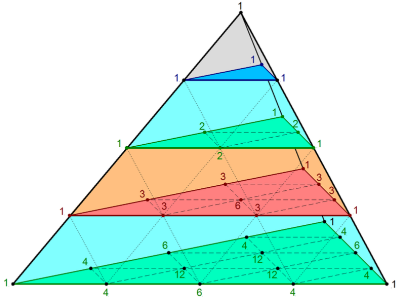




Multinomial Theorem.



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$(x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1+k_2+\cdots+x_r=n} \binom{n}{k_1,k_2,\ldots,k_r} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}.$

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"Stars and Bars" Proof.

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