Finite versus Infinite

We use functions to compare sizes of sets

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$$f' = (f - \{(m, n), (n, f(n))\}) \cup \{(m, f(n)), (n, n)\},$$

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Proof. Assume that $x \in A$ and $f(x) = C$. Then $x \in f(x)$ iff $x \notin f(x)$, a contradiction. □

Cantor's Theorem. For any set A, $|A| < |\mathcal{P}(A)|$.

Proof. By HW3, Problem 1, $F\colon A \to \mathcal{P}(A)\colon a \mapsto \{a\}$ is injective, so $|A| \leq |\mathcal{P}(A)|$. The theorem implies that $|A| \neq |\mathcal{P}(A)|$, so $|A| < |\mathcal{P}(A)|$. □

Exercise. Explain how Cantor's Theorem implies that $\mathcal{P}(\mathbb{N})$ is uncountable.