Practice Problems.

1. You roll four distinct 6-sided dice with sides numbered from 1-6 and sum the numbers obtained. What is the (classical) probability that the sum is 12?

The sample space for this problem is the set Ω of all outcomes obtained from the roll of four dice. An example of an outcome is $(2, 3, 5, 3)$. This sequence means we rolled 2 with the first die, 3 with the second and fourth dice, and 5 with the third die. The number of possible outcomes (= size of the sample space) is the number of such sequences, which is $6 \times 6 \times 6 \times 6 = 6^4$.

We are interested in computing the probability of the event E defined to be the set of all outcomes (x_1, x_2, x_3, x_4) such that $x_1 + x_2 + x_3 + x_4 = 12$. Our answer will then be $P(E) = |E|/|\Omega| = |E|/6^4$, so we need to compute $|E|$, which is the number of outcomes (x_1, x_2, x_3, x_4) such that $x_1 + x_2 + x_3 + x_4 = 12$.

I want to think of the problem of determining $|E|$ in terms of distributions of balls to boxes, therefore I will think of a single die as an empty box and a "roll of i" as the act of putting i identical balls into the box. From this perspective, the problem is to count the number of ways to put $k = 12$ identical balls into $n = 4$ distinct boxes subject to the restriction that the number of balls in each box is some number between 1 and 6. Let's count this by first counting how many ways there are to distribute $k = 12$ identical balls to $n = 4$ distinct boxes so that each box gets **at least one ball**, then subtract the number of distributions where some box gets **more than** 6 **balls**.

Let U be the set of distributions of 12 identical balls to 4 distinct boxes so that each box gets at least one ball. Let A_i, $i = 1, 2, 3, 4$, be the subset of distributions in U where Box i gets more than 6 balls. Since $|E| = |U| - |A_1 \cup A_2 \cup A_3 \cup A_4|$, I will evaluate $|U|$ first and evaluate $|A_1 \cup A_2 \cup A_3 \cup A_4|$ second.

The number of ways to distribute $k = 12$ identical balls to $n = 4$ distinct boxes so that each box gets at least one ball is $|U| = \left(\!\binom{n}{k-n}\!\right) = \left(\!\binom{4}{8}\!\right) = \binom{4+8-1}{8} = \binom{11}{8} = C(11, 8) = 165$. (Reference: see the Distributions Handout.)

Let's evaluate $|A_1 \cup A_2 \cup A_3 \cup A_4|$ where A_i is the subset of U consisting of those distributions where Box i receives > 6 balls. By the Inclusion/Exclusion formula,

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| &= \sum |A_i| - \sum |A_i \cap A_j| + \cdots \\ &= (|A_1| + |A_2| + |A_3| + |A_4|) - 0 + 0 - 0 + \text{ETC} \end{aligned}$$

Here some zeros appear because the sets A_i are pairwise disjoint ($A_i \cap A_j = \emptyset$ when $i \neq j$). The reason that this is true is that an element of $A_i \cap A_j$ is a distribution of 12 balls to 4 boxes where Box i gets at least 7 balls, Box j gets at least 7 balls, and the other boxes get at least 1 ball, and this would require at least $7 + 7 + 1 + 1 = 16$ balls, but we only have 12 balls. Thus,

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| = 4 \cdot |A_1|.$$

To compute $|A_1|$, first put 7 balls into Box 1 and 1 ball into each other box, thereby using up 10 balls. We must now distribute the remaining 2 balls to the four boxes subject to no restriction. The number of ways to distribute the remaining 2 identical balls to 4 boxes is $\left(\!\binom{2}{4}\!\right) = \binom{4+2-1}{2} = \binom{5}{2} = C(5,2) = 10$. This shows that $|A_1| = C(5,2) = 10 = |A_2| = |A_3| = |A_4|$.

To assemble our answer, we have shown that

$$|E| = |U| - |A_1 \cup A_2 \cup A_3 \cup A_4| = C(11,8) - 4 \cdot C(5,2) = 125.$$

The probability is therefore $|E|/|\Omega| = 125/6^4 = 0.096450617$.

2. You roll five dice that are shaped like the Platonic solids (Tetrahedron = 4-sided; Cube = 6-sided; Octahedron = 8-sided; Dodecahedron = 12-sided; Icosahedron = 20-sided). What is the probability that the sum is 15?

This is similar to the last problem.

Here $|\Omega| = 4 \times 6 \times 8 \times 12 \times 20 = 46080$.

Now let U be the set of distributions of 15 identical balls to 5 boxes where every box must get a ball. $|U| = \left(\!\binom{5}{15-5}\!\right) = \left(\!\binom{5}{10}\!\right) = C(5+10-1,10) = C(14,10) = 1001$.

Now let A_i be the subset of U consisting of those distributions where Box i receives $> s_i$ balls where $(s_1, s_2, s_3, s_4, s_5) = (4, 6, 8, 12, 20)$. One calculates that

(1) (a) $|A_1| = \left(\!\binom{5}{6}\!\right) = C(10,6) = 210$
  (b) $|A_2| = \left(\!\binom{5}{4}\!\right) = C(8,4) = 70$
  (c) $|A_3| = \left(\!\binom{5}{2}\!\right) = C(6,2) = 15$
  (d) $|A_4| = \left(\!\binom{5}{-2}\!\right) = 0$
  (e) $|A_5| = \left(\!\binom{5}{-10}\!\right) = 0$
(2) $|A_1 \cap A_2| = \left(\!\binom{5}{0}\!\right) = C(4,0) = 1$
(3) $|A_{i_1} \cap A_{i_2} \cap \cdots| = 0$ in all other cases.

Hence

$$|A_1 \cup \cdots \cup A_5| = \sum |A_i| - \sum |A_i \cap A_j| + \cdots = (210 + 70 + 15) - 1 = 294.$$

Hence $P(E) = |E|/|\Omega| = (1001 - 294)/46080 = 0.015342882$.

3. Rock-Paper-Scissors is a game played by two people. The players simultaneously form hand gestures in one of the three shapes: Rock (closed fist), Paper (flat hand), and Scissors (index and middle finger extended). If the two players make different gestures, then Rock beats Scissors, Scissors beats Paper, and Paper beats Rock. Otherwise, the game is a draw.

(a) What is the probability of a draw?

The sample space Ω is the set of pairs $(g_1, g_2) = (\text{gesture of player 1}, \text{gesture of player 2})$. There are $3 \times 3 = 9$ such pairs. The event we are interested in is the subset E of 'diagonal' pairs (g_1, g_1). There are three of these pairs. We compute that $P(E) = |E|/|\Omega| = 3/9 = 1/3$.

Now consider an n-player version, $n \geq 2$. In a given round of play, all players make simultaneous gestures. If exactly two different kinds of gestures are made, then those with the losing gesture are eliminated and the remaining players move to the next round where play is repeated. If it is not the case that exactly two different kinds of gestures are made in some round, then the round is a draw and all players proceed to the next round.

(b) What is the probability of a draw in the first round of the n-player version?

The sample space Ω is the set of n-tuples of gestures $(g_1, g_2, \ldots, g_n)$. There are $|\Omega| = 3^n$ such tuples. The event we are interested in is the subset E of n-tuples where exactly two different gestures appear. Let's find a way to evaluate $|E|$. (I will try to evaluate $|\Omega - E|$ first. This is the number of outcomes that are NOT draws.)

To count non-draws, fix one player 'Bob' from the n-players and choose a gesture g for Bob. This can be done in 3 ways (= 3 choices of a gesture for Bob). From the remaining $(n-1)$ players choose a proper subset of players to also make gesture g. This can be done in $2^{n-1} - 1$ ways ($= 2^{n-1} - 1$ proper subsets of $(n-1)$ people). Now choose a different gesture g' for the remaining players. This can be done in 2 ways (= 2 choices for g'). Altogether, we find that $|\Omega - E| = 3 \cdot 2 \cdot (2^{n-1} - 1) = 3(2^n - 2)$, hence $|E| = |\Omega| - |\Omega - E| = 3^n - 3(2^n - 2)$. We compute that $P(E) = |E|/|\Omega| = (3^n - 3(2^n - 2))/3^n$.

4. A genii has the power to grant 100 wishes of identical magical content. The genii intends to distribute these wishes to 10 Discrete Mathematics students. What is the probability that each student will get at least 3 wishes if the genii is required to give away all of his wishes?

The sample space Ω is the set of distributions of 100 identical wishes to 10 distinct recipients. From the Distributions Handout, $|\Omega| = \left(\!\binom{10}{100}\!\right) = C(109, 100)$.

The event of interest is the set E of all distributions where each student gets at least 3 wishes. To evaluate $|E|$, imagine first giving 3 wishes each to the 10 students, then distributing the remaining 70 wishes arbitrarily. From the Distributions Handout, the number of ways to distribute the remaining 70 wishes is $|E| = \left(\!\binom{10}{70}\!\right) = C(79, 70)$.

The desired probability is $P(E) = \left(\!\binom{10}{70}\!\right) / \left(\!\binom{10}{100}\!\right) = \frac{79!100!}{70!109!} = 0.04827379$.

5. If $f : k \to k$ is a permutation, then i is called a **fixed point** of f if $f(i) = i$. What is the probability that a randomly chosen permutation $f : k \to k$ has no fixed points? What happens to your answer as $k \to \infty$?

The sample space Ω is the set of all permutations $f\colon k \to k$. $|\Omega| = k!$.

Now, for $i = 1 \ldots, k$, let A_i be the subset of Ω consisting of the bijections where $f(i) = i$. (That is, $f \in A_i$ if and only if f has i as a fixed point.) The event of interest is the set of permutations with no fixed points, i.e., $E = \Omega - (\bigcup A_i)$. To evaluate $|E| = |\Omega| - |\bigcup A_i| = k! - |\bigcup A_i|$ we apply inclusion/exclusion to evaluate $|\bigcup A_i|$.

$$\begin{aligned} |E| &= k! - \left(\binom{k}{1}(k-1)! - \binom{k}{2}(k-2)! + \cdots\right) \\ &= k!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^k}{k!}\right). \end{aligned}$$

The desired probability is $P(E) = |E|/|\Omega| = |E|/k! = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^k}{k!}$. Observe that this probability approaches $\frac{1}{e}$ as $k \to \infty$.