Practice Problems About Russell's Paradox and the Axiom of Foundation.

(1) (Russell's pair of ducks.) Bertrand Russell gives a pair of ducks to those people, and only those people, who do not give a pair of ducks to themselves. Explain why

(a) Russell cannot fail to give himself a pair of ducks.

Suppose that Russell DID fail to give himself a pair of ducks. Then, since he is one of the people who did not give himself a pair of ducks, Russell must give this person (=himself) a pair of ducks. Thus, if Russell does not give himself a pair of ducks, then he must give himself a pair of ducks.

(b) It is not possible for Russell to give himself a pair of ducks.

Suppose that Russell DOES give himself a pair of ducks. Then he is one of those people who give a pair of ducks to themselves. According to the rules, Russell does not give this person (=himself) a pair of ducks. Thus, if Russell does give himself a pair of ducks, then he does not give himself a pair of ducks.

(2) Show that the union of all sets is not a set.

Let $\mathcal{S}$ be the proper class of all sets. $\bigcup \mathcal{S} = \mathcal{S}$, so the union of $\mathcal{S}$ is a proper class and not a set.

(3) Find a sequence of sets satisfying $\cdots A_3 \subsetneq A_2 \subsetneq A_1 \subsetneq A_0$.

$$\cdots \subsetneq \{3, 4, \ldots\} \subsetneq \{2, 3, 4, \ldots\} \subsetneq \{1, 2, 3, 4, \ldots\} \subsetneq \{0, 1, 2, 3, 4, \ldots\}$$

which may be written

$$\cdots \subsetneq \mathbb{N} \setminus \{0, 1, 2\} \subsetneq \mathbb{N} \setminus \{0, 1\} \subsetneq \mathbb{N} \setminus \{0\} \subsetneq \mathbb{N}$$

Here I am using the set subtraction symbol: $A \setminus B$, which is used to express $\{x \in A \mid x \notin B\}$. Some people write $A - B$ for $A \setminus B$.

(4) Find a sequence of sets satisfying $B_0 \in B_1 \in B_2 \in B_3 \in \cdots$.

$$\emptyset \in \mathcal{P}(\emptyset) \in \mathcal{P}\mathcal{P}(\emptyset) \in \mathcal{P}\mathcal{P}\mathcal{P}(\emptyset) \in \cdots$$

or

$$0 \in 1 \in 2 \in 3 \in 4 \in \cdots$$

(5) Find all 3-element sets A such that $A \subseteq \mathcal{P}(A)$. (There are only 2 of them.)

Using the Axiom of Foundation, one can show that the only possibilities are

$$A = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \qquad A' = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}.$$

(First show that if $A \subseteq \mathcal{P}(A)$ and $A = \{E_1, E_2, E_3\}$ has an $\in$-minimal element, E_1, then it must be that $E_1 = \emptyset$. Now delete this element to obtain $\{E_2, E_3\} = A \setminus \{E_1\}$ where $E_2 \neq \emptyset \neq E_3$. If $\{E_2, E_3\}$ has an $\in$-minimal element, E_2, then the elements of E_2 must belong to $\{E_1\}$, hence $E_2 = \{E_1\} = \{\emptyset\}$. Continue this process to deduce that the elements of E_3 must belong to $\{E_1, E_2\}$, ETC.)

Sets A satisfying $A \subseteq \mathcal{P}(A)$ are called *transitive*. Some examples of transitive sets are: $0, 1, 2, 3, \ldots$ and $\mathbb{N}$.