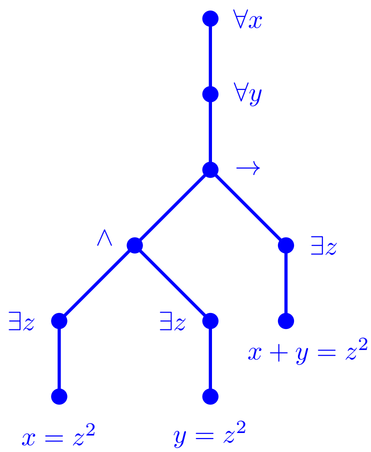
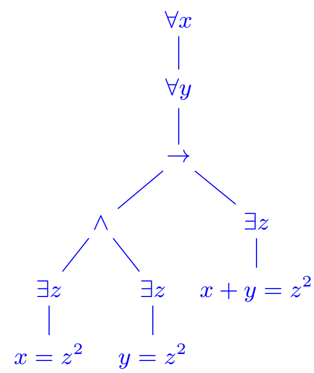
Solutions to HW 7.

1. This problem concerns the formal sentence

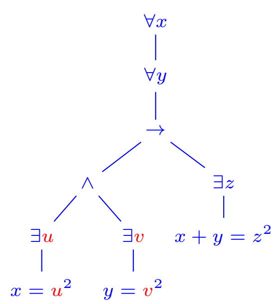
$$(\forall x)(\forall y)((((\exists z)(x = z^2)) \wedge ((\exists z)(y = z^2))) \rightarrow ((\exists z)(x + y = z^2))).$$

   (a) Draw the formula tree for this sentence.
   [Two Latex versions! The first uses "tikz" and the second uses the "forest" package.]

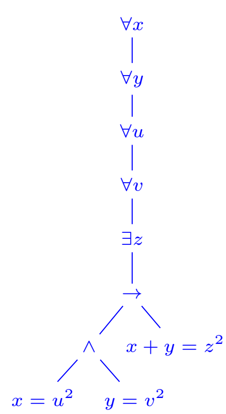




   (b) Standardize the variables apart.



   (c) Write the sentence in prenex form.



$$(\forall x)(\forall y)(\forall u)(\forall v)(\exists z)(((x = u^2) \wedge (y = v^2)) \to (x + y = z^2))$$

2. This problem also concerns the formal sentence from Problem 1.
   (a) Is the sentence true in the natural numbers, $\mathbb{N}$? No!
   Give a winning strategy for the appropriate quantifier. (Appropriate quantifier is $\forall$.)
   - $\forall$ chooses $x = 1$.
   - $\forall$ chooses $y = 1$.
   - $\forall$ chooses $u = 1$.
   - $\forall$ chooses $v = 1$.
   - To win, $\exists$ would have to choose z so that $z^2 = 2$. There is no such $z \in \mathbb{N}$, so $\exists$ loses.

   (b) Is the sentence true in the real numbers, $\mathbb{R}$? Yes!
   Give a winning strategy for the appropriate quantifier. (Appropriate quantifier is $\exists$.)
   - $\forall$ chooses any x.
   - $\forall$ chooses any y.
   - $\forall$ chooses any u.
   - $\forall$ chooses any v.

   $\forall$ has already lost, unless the choices made satisfy $x = u^2$ and $y = v^2$, so assume that these equalities hold.
   - To win, $\exists$ would have to choose z so that $z^2 = x + y = u^2 + v^2$. So $\exists$ can win by choosing $z = \sqrt{u^2 + v^2}$. This choice is possible, since in $\mathbb{R}$ squares are nonnegative, so $u^2, v^2 \geq 0$. Also, in $\mathbb{R}$, a sum of nonnegative numbers is nonnegative, from which we get $u^2 + v^2 \geq 0$. Finally, any nonnegative real number has a real square root, so it is possible to choose a real number z satisfying $z = \sqrt{u^2 + v^2}$.

3. Negate the sentence from Problem 1 and then rewrite the negation so that it is in prenex form.

$$\neg(\forall x)\ (\forall y)\ (\forall u)\ (\forall v)\ (\exists z)\ (((x = u^2) \wedge (y = v^2)) \to (x + y = z^2)).$$

$$(\exists x)\ (\exists y)\ (\exists u)\ (\exists v)\ (\forall z)\ \neg(((x = u^2) \wedge (y = v^2)) \to (x + y = z^2)).$$

You can take this further, if you choose:

$$(\exists x)\ (\exists y)\ (\exists u)\ (\exists v)\ (\forall z)\ (((x = u^2) \wedge (y = v^2)) \wedge \neg(x + y = z^2)).$$