Solutions to HW 6.

1. Determine whether the negation of the proposition P logically implies, is logically equivalent to, or is logically independent of proposition Q:

(i) $P = a \to b, Q = a \wedge (\neg b)$

The negation of P is logically equivalent to Q. We can verify this with truth tables.

a	b	$P = a \to b$	$\neg P = \neg(a \to b)$	$\neg b$	$Q = a \wedge (\neg b)$
0	0	1	0	1	0
0	1	1	0	0	0
1	0	0	1	1	1
1	1	1	0	0	0

Note: This is an important example. If we want to justify the claim that an implication might be false (i.e., $P = a \to b$ does not hold for certain a and b), then we should explain why it is possible for the hypothesis "a" to be true and the conclusion "b" to be false (i.e., that $Q = a \wedge (\neg b)$ holds). This problem shows that establishing that $a \to b$ is false is equivalent to establishing that $a \wedge (\neg b)$ is true.

There is another way to do this problem using equivalences that we have already established.

$$
\begin{aligned}
\neg P &= \neg(a \to b) && \text{(From the Definition of } P\text{)} \\
&\equiv \neg((\neg a) \vee b) && \text{(``}\to\text{ is redundant'')} \\
&\equiv (\neg(\neg a)) \wedge (\neg b) && \text{(De Morgan's Law)} \\
&\equiv a \wedge (\neg b) && \text{(Law of Double Negation)} \\
&= Q && \text{(Definition of } Q\text{)}
\end{aligned}
$$

(ii) $P = (a \to b) \to a$, $Q = \neg a$.

The negation of P is logically equivalent to Q. We can verify this with truth tables.

a	b	$a \to b$	$P = (a \to b) \to a$	$\neg P$	$Q = \neg a$
0	0	1	0	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	1	0	0

(iii) $P = (a \to b) \wedge (b \to c)$, $Q = (a \to c)$.

Let's compare the truth tables of $\neg P$ and Q.

a	b	c	$X = a \to b$	$Y = b \to c$	$P = X \wedge Y$	$\neg P$	$Q = a \to c$
0	0	0	1	1	1	0	1
0	0	1	1	1	1	0	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	0	1
1	0	0	0	1	0	1	0
1	0	1	0	1	0	1	1
1	1	0	1	0	0	1	0
1	1	1	1	1	1	0	1

This is enough information to conclude that $\neg P$ and Q are logically independent. We see that is is possible for $\neg P$ to be true when Q is false (line 5 or line 7), so $\neg P$ does not logically imply Q. It is also possible for Q to be true when $\neg P$ is false (lines 1, 2, 4 or 8), so Q does not logically imply $\neg P$. (These facts can also be expressed by saying that $(\neg P) \to Q$ is not a tautology and $Q \to (\neg P)$ is not a tautology.)

Side comment: The table shows that P logically implies Q. This establishes the transitivity of implication. That is, $(a \to b) \wedge (b \to c)$ logically implies $(a \to c)$.

2. Write the following propositions in disjunctive normal form, assuming that each proposition is a function of p, q and r.

(i) $p \to r$

$$\underline{((\neg p) \wedge (\neg q) \wedge (\neg r))} \vee \underline{((\neg p) \wedge (\neg q) \wedge r)} \vee \underline{((\neg p) \wedge q \wedge (\neg r))} \vee \underline{((\neg p) \wedge q \wedge r)} \vee \underline{(p \wedge (\neg q) \wedge r)} \vee \underline{(p \wedge q \wedge r)}$$

(ii) $((p \to q) \to ((\neg p) \leftrightarrow r))$.

$$\underline{((\neg p) \wedge (\neg q) \wedge r)} \vee \underline{((\neg p) \wedge q \wedge r)} \vee \underline{(p \wedge (\neg q) \wedge (\neg r))} \vee \underline{(p \wedge (\neg q) \wedge r)} \vee \underline{(p \wedge q \wedge (\neg r))}$$

(iii) q

$$\underline{((\neg p) \wedge q \wedge (\neg r))} \vee \underline{((\neg p) \wedge q \wedge r)} \vee \underline{(p \wedge q \wedge (\neg r))} \vee \underline{(p \wedge q \wedge r)}$$

3. Write the following axioms of set theory as formal sentences.

(i) Extensionality.

$$(\forall A)(\forall B)((A = B) \leftrightarrow (\forall z)((z \in A) \leftrightarrow (z \in B)))$$

(ii) Pairing.

$$(\forall A)(\forall B)(\exists P)(\forall z)((z \in P) \leftrightarrow ((z = A) \vee (z = B)))$$

(iii) Power set.

$$(\forall A)(\exists P)(\forall z)((z \in P) \leftrightarrow \underline{(z \subseteq A)})$$

or

$$(\forall A)(\exists P)(\forall z)((z \in P) \leftrightarrow \underline{((\forall w)((w \in z) \to (w \in A)))})$$