Counting

Two basic counting principles

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In general, the Sum Rule is suggested when (exclusive) "OR" is being counted, while the Product Rule is suggested when (independent) "AND" is being counted.

Size of a power set using the Sum Rule

Theorem.

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Let $A \subseteq \mathcal{P}(X_{n+1})$ be the set of those subsets of X that do not contain the last element, x_{n+1}, and let B be the set of those subsets of X that do contain the last element, x_{n+1}.

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Size of a power set using the Product Rule

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Proof. We count the number of subsets of X by counting the number of "descriptions" of subsets. This means we will count the number of characteristic functions $c\colon X \to \{0, 1\}$. A subset $S = \{2, 3, 5\}$ of $X = \{1, 2, 3, \ldots, n\}$ may be "described" by its characteristic function:

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x	1	2	3	4	5	$\cdots$	n
$c(x)$	0	1	1	0	1	$\cdots$	0

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x	1	2	3	4	5	$\cdots$	n
$c(x)$	0	1	1	0	1	$\cdots$	0

There are 2 choices for $c(1)$, 2 independent choices for $c(2)$, $\ldots$, 2 independent choices for $c(n)$, so $|X| = 2 \cdot 2 \cdots 2 = 2^n$.

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Here we used the fact that a subset $S \subseteq X$ can be described by specifying whether $1 \in S$ AND specifying whether $2 \in X$, etc.

Exercise! Colorado license plates

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From 2000-2015, CO license plates followed the pattern 000-AAA.

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   $10 \times 10 \times 10 \times 26 \times 26 \times 26 = 10^3 \cdot 26^3$

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   $10 \times 10 \times 10 \times 26 \times 26 \times 26 = 10^3 \cdot 26^3 = 17,576,000$

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   $10^3 \cdot 26^3 + 26^3 \cdot 10^3 + 26^4 \cdot 10^2 = 80,849,600$

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Definition. A bijection $b\colon X \to X$ from a set X to itself is called a **permutation** of X.

Exercise! Create poetry with math!

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Roses are red.
Violets are blue.

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