Conditional Probability

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We write $P(E|F)$ to mean "the probability of event E GIVEN that event F occurs".

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Practice problems

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1. Suppose that Alice draws a 13-card bridge hand from a standard deck of 52 cards and gets two aces. Next, Bob draws a 13-card hand from the remaining cards. What is the probability that Bob gets one ace? Does this differ from the probability that Bob gets one ace when drawing 13 cards from a fresh deck?

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2. Let $\Omega = \{1, 2, \ldots, 15\}$, $E = \{\text{primes in } \Omega\}$, $F = \{\text{odd numbers in } \Omega\}$. Compute $P(E)$, $P(F)$, $P(E \cap F)$, $P(E|F)$, $P(F|E)$. Are E and F independent events?

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3. Suppose that you roll two distinct 6-sided dice. Let E be the event that the first die shows a 3 and let F be the event that the sum of the numbers shown is 8. Are E and F independent?

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3. Suppose that you roll two distinct 6-sided dice. Let E be the event that the first die shows a 3 and let F be the event that the sum of the numbers shown is 8. Are E and F independent?

4. Explain why, if $F \subseteq E$, it is the case that $P(E|F) = 1$. Explain why, if E and F are mutually exclusive events, it is the case that $P(E|F) = 0$.

Solution sketches! (in reverse order), # 4

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Explain why, if $F \subseteq E$, it is the case that $P(E|F) = 1$. Explain why, if E and F are mutually exclusive events, it is the case that $P(E|F) = 0$.

If $F \subseteq E$, then $E \cap F = F$, so

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1.$$

If E and F are mutually exclusive, then $E \cap F = \emptyset$, so

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(\emptyset)}{P(F)} = 0.$$

For the last equality, I am using that $P(\emptyset) = 0$. This fact follows from the Kolmogorov Axioms. Namely,

$$1 = P(\Omega) = P(\Omega \cup \emptyset) = P(\Omega) + P(\emptyset) = 1 + P(\emptyset),$$

so $P(\emptyset) = 0$.

Solution sketches! (in reverse order), # 3

Solution sketches! (in reverse order), # 3

Suppose that you roll two distinct 6-sided dice. Let E be the event that the first die shows a 3 and let F be the event that the sum of the numbers shown is 8. Are E and F independent?

Verify that $P(E) = \frac{1}{6}$, $P(F) = \frac{5}{36}$, and $P(E \cap F) = \frac{1}{36}$. Therefore

$$P(E)P(F) = \frac{1}{6} \cdot \frac{5}{36} \neq \frac{1}{36} = P(E \cap F).$$

It follows that E and F are NOT independent.

Solution sketches! (in reverse order), # 2

Solution sketches! (in reverse order), # 2

Let $\Omega = \{1, 2, \ldots, 15\}$, $E = \{\text{primes in } \Omega\}$, $F = \{\text{odd numbers in } \Omega\}$. Compute $P(E), P(F), P(E \cap F), P(E|F), P(F|E)$. Are E and F independent events?

Answers.
$P(E) = \frac{6}{15}$, $P(F) = \frac{8}{15}$, $P(E \cap F) = \frac{5}{15}$, $P(E|F) = \frac{5}{8}$, $P(F|E) = \frac{5}{6}$.

Since $P(E) \neq P(E|F)$, E and F are not independent.

Solution sketches! (in reverse order), # 1

Solution sketches! (in reverse order), # 1

Suppose that Alice draws a 13-card bridge hand from a standard deck of 52 cards and gets two aces. Next, Bob draws a 13-card hand from the remaining cards. What is the probability that Bob gets one ace? Does this differ from the probability that Bob gets one ace when drawing 13 cards from a fresh deck?

Let A be the event that Alice draws two aces and let B be the event that Bob draws one ace from the remaining cards given that A occurs. Verify the following:

$$P(A) = \frac{\binom{4}{2} \cdot \binom{48}{11}}{\binom{52}{13}}, \qquad P(B|A) = \frac{\binom{2}{1} \cdot \binom{37}{12}}{\binom{39}{13}}.$$

On the other hand, let C be the event that Bob gets one ace when drawing 13 cards from a fresh deck. Now we have

$$P(C) = \frac{\binom{4}{1} \cdot \binom{48}{12}}{\binom{52}{13}}.$$

Observe that $P(B|A) = 0.456140351 > 0.438847539 = P(C)$, so the probabilities differ.