The Axioms of Replacement, Choice, and Foundation

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To remember: If $F(x)$ is a class function and A is a set, then $F(A)$ (the image of A under F) is a set.

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Bertrand Russell highlighted the nonconstructive nature of this axiom when he wrote:

The Axiom of Choice is necessary to select a set from an infinite number of pairs of socks, but not an infinite number of pairs of shoes.

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An enumeration of $\{a, b, c\}$ is a bijection $e\colon 3 \to \{a, b, c\}$. An enumeration of the set of prime numbers is $p\colon \mathbb{N} \to \text{Primes}\colon n \mapsto p_n = n$th prime.

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If $x_0 \ni x_1 \ni x_2 \ni \cdots$ looks like an infinite descending $\in$-chain, then $A = \{x_0, x_1, \ldots\}$ has no $\in$-minimal element. Conversely, if $A \neq \emptyset$ has no $\in$-minimal element, then one can recursively construct an infinite descending $\in$-chain.

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