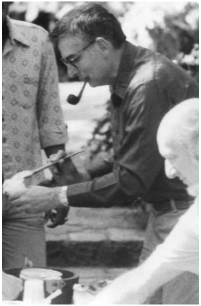
When $1 < I(T, \aleph_0) < \aleph_0$



Theories with $I(T, \aleph_0) = n$

Throughout these slides, T will be a complete theory in a countable language. Usually, we will also assume that T has infinite models.

It is easy to construct a complete theory in a countable language that has $n = 0$ isomorphism types of countably infinite models. E.g. $\mathrm{Th}(\mathbf{A})$ for some finite $\mathbf{A}$.

A theory in a countable language that has no finite maodels and has $n = 1$ isomorphism type of countably infinite model must be complete (Łos-Vaught Completeness Criterion). These are the ω-categorical theories. They were characterized by Engeler, Ryll-Nardzewski, and Svenonius.

We know some examples of ω-categorical structures (infinite pure set, $\langle \mathbf{Q}; < \rangle$, the random graph, $\aleph_0$-dimensional vector spaces over finite fields). Moreover, once we know some we can generate others:

Theorem. If $\mathbf{A}$ and $\mathbf{B}$ have the same language and both have oligomorphic automorphism groups, then $\mathbf{A} \times \mathbf{B}$ has an oligomorphic automorphism group. If the language is relational, then the same holds for the disjoint union $\mathbf{A} \sqcup \mathbf{B}$.

What if $n > 1$?

It is easy to construct incomplete theories with n models for any finite n.

Consider the language L with n unary predicates (colors of elements) and let T be the theory of infinite monochromatic sets. (I.e., any model of T is infinite and realizes exactly one of the n colors.) For $n = 3$, this theory is axiomatized by sentences saying that models are infinite, along with

$$(\exists x)\text{Red}(x) \rightarrow ((\forall x)\text{Red}(x) \wedge \neg(\exists x)\text{Blue}(x) \wedge \neg(\exists x)\text{Green}(x))$$

plus all other sentences obtained from this one by permuting the colors.

Ehrenfeucht's Example shows that there are complete theories with $n = 3, 4, 5, \ldots$ many isomorphism types of countable models.

What if $n = 2$ and T is complete?

Theorem. (Vaught) If T is a complete theory in a countable language L, then $I(T, \aleph_0) \neq 2$.

So, 2 is the only finite number that cannot arise as $I(T, \aleph_0)$ for complete T in a countable language.

One might call this theorem "Vaught's Theorem on the Nonexistence of a Complete Theory in a Countable Language with Exactly Two isomorphism Types of Countable Models".

Some people call it "Vaught's *Never Two* Theorem".
(A Google search using keywords "Vaught" and "never two" yielded 579,000 hits on April 28, 2024.)

It is proved by showing that if $1 < I(T, \aleph_0) \leq \aleph_0$, then T has an atomic model, $\mathbf{A}$, a nonisomorphic ω-saturated model, $\mathbf{S}$, and a model $\mathbf{B}$ that is neither atomic nor ω-saturated.

Never Two: The Proof

Stage 1: T is small, hence has a countably infinite ω-saturated model $\mathbf{S}$.

Stage 2: T must also have a countably infinite atomic model $\mathbf{A}$, and $\mathbf{A} \ncong \mathbf{S}$.

Stage 3: There must exist $\mathbf{a} \in \mathbf{S}^n$ whose type is not isolated.

Stage 4: The expansion $\mathbf{S_a}$ is an ω-saturated model of $T_\mathbf{a} := \mathrm{Th}(\mathbf{S_a})$, which is a small theory. (The theory of any weakly saturated model is small.) Hence $T_\mathbf{a}$ has a countably infinite atomic model $\mathbf{B_b}$.

Stage 5: $\mathrm{Aut}(\mathbf{S})$ does not act oligomorphically on $\mathbf{S}$, so the subgroup $\mathrm{Aut}(\mathbf{S_a})$ cannot act oligomorphically on $\mathbf{S_a}$. Hence $T_\mathbf{a}$ is not ω-categorical. Hence $\mathbf{S_a} \ncong \mathbf{B_b}$. Hence $\mathbf{B_b}$ is not an ω-saturated model of $T_\mathbf{a}$.

Stage 6: The reduct $\mathbf{B}$ of $\mathbf{B_b}$ to L cannot be isomorphic to $\mathbf{A}$, since the type of $\mathbf{b} \in \mathbf{B}^n$ is nonisolated.

Stage 7: The reduct $\mathbf{B}$ of $\mathbf{B_b}$ to L also cannot be isomorphic to $\mathbf{S}$. Since $\mathbf{B_b}$ is not an ω-saturated model of $T_\mathbf{a}$. $\square$

The number of isomorphism types of countable models

We have seen examples of complete theories T in a countable language where the number $I(T, \aleph_0)$ of isomorphism types of countable models is any of the following cardinals.

1. 0 (Th($\mathbf{A}$), $\mathbf{A}$ finite.)
2. 1 (Theory of an infinite pure set, of ordered set $\langle \mathbb{Q}; < \rangle$, of the random graph, of an infinite-dimensional vector space over a finite field).
3. 2 (Whoops! “Never two”.)
4. $3, 4, 5, \ldots$ (Ehrenfeucht-type theories.)
5. $\aleph_0$ (ACF$_p$, infinite $\mathbb{Q}$-vector spaces.)
6. $2^{\aleph_0}$ (Theory of $\langle \omega; +, \cdot \rangle$, of the field $\mathbb{Q}$, any model of ZFC.)

Vaught's Conjecture

Vaught conjectured that the cardinals on the previous slide are the only possibilities. Specifically,

Vaught's Conjecture, (1961). If T is a complete theory in a countable language and T has uncountably many isomorphism types of countable models, then T has continuumly many isomorphism types of countable models. (Equivalently, $\aleph_0 < I(T, \aleph_0) < 2^{\aleph_0}$ is impossible.)

This was originally stated in the form of a problem rather than a conjecture.

If CH holds ($\aleph_0^+ = 2^{\aleph_0}$), then there is nothing to prove, so why bother?

Reformulated Conjecture. If T is a complete theory in a countable language and all type spaces $S_n(T)$ are scattered, then T has countably many isomorphism types of countable models.

Morley's Theorem. If T is a complete theory in a countable language, then

1. if all $S_n(T)$ are scattered, then $I(T, \aleph_0) \leq \aleph_1$,
2. hence if $\aleph_0 < I(T, \aleph_0) < 2^{\aleph_0}$, then $I(T, \aleph_0) = \aleph_1$.

Vaught's Conjecture is that:

1. if all $S_n(T)$ are scattered, then $I(T, \aleph_0) \leq \aleph_0$,
2. hence $\aleph_0 < I(T, \aleph_0) < 2^{\aleph_0}$ is impossible.

In 2002, Robin Knight of Oxford posted a 117-page preprint with a "proposed counterexample" to Vaught's Conjecture. He asserts to have constructed a complete theory T in a countable language with $I(T, \aleph_0) = \aleph_1$. The current status of this counterexample is "not verified".

The Baldwin-Lachlan Theorem

For any uncountably categorical theory there are a limited number of possibilities for $I(T, \aleph_0)$.

Theorem. If T is a complete theory in a countable language, and κ is uncountable, then $I(T, \kappa) = 1$ implies that $I(T, \aleph_0) = 1$ or $I(T, \aleph_0) = \aleph_0$.

In particular, if $S_n(T)$ is uncountable for some n, then T is not κ-categorical for any infinite κ.

A related result:

Morley's Other Theorem. If T is a complete theory in a countable language, and $\kappa \leq \kappa'$ are uncountable, then $I(T, \kappa) = 1$ if and only if $I(T, \kappa') = 1$.

Problem for you! Can you give examples of complete, countable theories T for which the pair $(I(T, \aleph_0), I(T, \aleph_1))$ realizes each of the possibilities $(1, 1)$, $(1, \neq 1)$, $(\neq 1, 1)$, $(\neq 1, \neq 1)$?

Some answers to the Problem on the previous slide

1. (Complete theories that are κ-categorical for all infinite κ) The theory of infinite sets. The theory of infinite $\mathbb{F}_q$-vector spaces. The theory of infinite sets equipped with a fixed-point-free permutation of prime order.

2. (Complete theories that are categorical in power $\aleph_0$ only) DLO. The theory of the random graph. The theory of an equivalence relation with exactly two classes, both infinite.

3. (Complete theories that are κ-categorical for uncountable κ only) ACF_p. The theory of infinite $\mathbb{Q}$-vector spaces. The theory of the natural numbers equipped with the successor function. The theory of infinite, 2-regular, acyclic graphs.

4. (Complete theories that are not categorical in any infinite power) Any complete theory where $S_n(T)$ is uncountable for some n. For example, the complete theory of $\langle \omega; +, \cdot \rangle$. The complete theory of any model of ZFC. The theory of countably many independent unary predicates.