When $1 < I(T, \aleph_0) < \aleph_0$

Theories with finitely many isotypes of countable models

Throughout these slides, *T* will be a complete theory in a countable language which has infinite models.

It is easy to construct a complete theory in a countable language that has no infinite models. E.g. Th($\mathbb{A}$) for some finite $\mathbb{A}$.

A theory in a countable language that has exactly one isotype of countably infinite model must be complete (Łos-Vaught Completeness Criterion). These are the ω-categorical theories. They were characterized by Engeler, Ryll-Nardzewski, and Svenonius.

We know some examples of ω-categorical structures (infinite pure set, $\langle \mathbb{Q}; < \rangle$, the random graph, $\aleph_0$-dimensional vector spaces over finite fields). Moreover, once we know some we can generate others:

Theorem. If $\mathbb{A}$ and $\mathbb{B}$ have the same language and both have oligomorphic automorphism groups, then $\mathbb{A} \times \mathbb{B}$ has an oligomorphic automorphism group. If the language is relational, then the same holds for the disjoint union $\mathbb{A} \sqcup \mathbb{B}$.

What if $n > 1$?

It is easy to construct <u>incomplete</u> theories with n models for any finite n.

Consider the language L with n unary predicates (colors of elements) and let T be the theory of infinite monochromatic sets. (I.e., any model of T realizes exactly one of the n colors.) For $n = 3$, this theory is axiomatized by sentences saying that models are infinite, along with

$$(\exists x)\text{Red}(x) \rightarrow ((\forall x)\text{Red}(x) \wedge \neg(\exists x)\text{Blue}(x) \wedge \neg(\exists x)\text{Green}(x))$$

plus all other sentences obtained from this one by permuting the colors.

Ehrenfeucht's Example shows that there are <u>complete</u> theories with $n = 3, 4, 5, \ldots$ many isomorphism types of countable models.

What if $n = 2$?

Theorem. (Vaught) If T is a complete theory in a countable language L, then $I(T, \aleph_0) \neq 2$.

So, 2 is the only finite number that cannot arise as $I(T, \aleph_0)$ for complete T in a countable language.

One might call this theorem "Vaught's Theorem on the Nonexistence of a Complete Theory in a Countable Language with Exactly Two isomorphism Types of Countable Models".

I have heard this theorem called "Vaught's *Never Two* Theorem".

It is proved by showing that if $1 < I(T, \aleph_0) \leq \aleph_0$, then T has a model $\mathbb{B}$ that is neither atomic nor ω-saturated.

Never Two: The Proof

Stage 1: T is small, hence has a countably infinite ω-saturated model $\mathbb{S}$.

Stage 2: T must also have a countably infinite atomic model $\mathbb{A}$, and $\mathbb{A} \not\cong \mathbb{S}$.

Stage 3: There must exist $\mathbf{a} \in \mathbb{S}^n$ whose type is not isolated.

Stage 4: The expansion $\mathbb{S}_{\mathbf{a}}$ is an ω-saturated model of $T_{\mathbf{a}} := \mathrm{Th}(\mathbb{S}_{\mathbf{a}})$, which is a small theory. Hence $T_{\mathbf{a}}$ has a countably infinite atomic model $\mathbb{B}_{\mathbf{b}}$.

Stage 5: $\mathrm{Aut}(\mathbb{S})$ does not act oligomorphically on $\mathbb{S}$, so the subgroup $\mathrm{Aut}(\mathbb{S}_{\mathbf{a}})$ cannot act oligomorphically on $\mathbb{S}_{\mathbf{a}}$. Hence $\mathbb{S}_{\mathbf{a}} \not\cong \mathbb{B}_{\mathbf{b}}$.

Stage 6: The reduct $\mathbb{B}$ of $\mathbb{B}_{\mathbf{b}}$ to L cannot be isomorphic to $\mathbb{A}$.

Stage 7: The reduct $\mathbb{B}$ of $\mathbb{B}_{\mathbf{b}}$ to L also cannot be isomorphic to $\mathbb{S}$. $\square$

The number of isomorphism types of countable models

We have seen examples of complete theories T in a countable language where the number $I(T, \aleph_0)$ of isomorphism types of countable models is any of the following cardinals.

1. 0 (Th($\mathbb{A}$), $\mathbb{A}$ finite.)
2. 1 (Theory of an infinite pure set, of ordered set $\langle \mathbb{Q}; < \rangle$, of the random graph, of an infinite-dimensional vector space over a finite field).
3. 2 (Whoops! "Never two".)
4. $3, 4, 5, \ldots$ (Ehrenfeucht-type theories.)
5. $\aleph_0$ (ACF$_0$, infinite $\mathbb{Q}$-vector spaces.)
6. $2^{\aleph_0}$ (Theory of $\langle \omega; +, \cdot \rangle$, of field $\mathbb{Q}$, any complete extension of ZFC.)

Vaught's Conjecture

Vaught conjectured that the cardinals on the previous slide are the only possibilities. Specifically,

Conjecture. If T is a complete theory in a countable language and T has uncountably many isomorphism types of countable models, then T has continuumly many isomorphism types of countable models. (Equivalently, $\aleph_0 < I(T, \aleph_0) < 2^{\aleph_0}$ is impossible.)

If CH holds ($\aleph_0^+ = 2^{\aleph_0}$), then there is nothing to prove, so why bother?

Reformulated Conjecture. If T is a complete theory in a countable language and all type spaces $S_n(T)$ are scattered, then T has countably many isomorphism types of countable models.

Morley's Theorem

Theorem. If T is a complete theory in a countable language and $\aleph_0 < I(T, \aleph_0) < 2^{\aleph_0}$, then $I(T, \aleph_0) = \aleph_1$.

That is, Morley proved that if all $S_n(T)$ are scattered, then $I(T, \aleph_0) \leq \aleph_1$. Vaught's Conjecture is the stronger inequality $I(T, \aleph_0) \leq \aleph_0$.

In 2002, Robin Knight of Oxford posted a 117-page preprint with a "proposed counterexample" to Vaught's Conjecture. He asserts to have constructed a theory T with $I(T, \aleph_0) = \aleph_1$. The current status of this counterexample seems to be "not verified".