Theorem 1. *(Ultrafilter Convergence Theorem) Let $\mathcal{X} = \langle X; \mathcal{T} \rangle$ be a topological space.*

(1) *$\mathcal{X}$ is compact iff for every ultrafilter $\mathcal{U}$ on a set I and every function $f\colon I \to X$ there is at least one $\ell \in X$ such that $\lim_{\mathcal{U}} f = \ell$.*

(2) *$\mathcal{X}$ is Hausdorff iff for every ultrafilter $\mathcal{U}$ on a set I and every function $f\colon I \to X$ there is at most one $\ell \in X$ such that $\lim_{\mathcal{U}} f = \ell$.*

Proof. [(1), $\Rightarrow$] Assume that $\mathcal{X}$ is compact, but there is some ultrafilter $\mathcal{U}$ on some set I and some function $f\colon I \to X$ with no ℓ satisfying $\lim_{\mathcal{U}} f = \ell$. Each $x \in X$ must have an open neighborhood O_x such that $f^{-1}(O_x) \notin \mathcal{U}$. By compactness, the open cover $\{O_x \mid x \in X\}$ has a finite subcover $\{\mathcal{O}_1, \ldots, O_n\}$. This yields $\cup_{i=1}^n f^{-1}(O_i) = f^{-1}(\cup_{i=1}^n O_i) = I \in \mathcal{U}$. We claim that if a finite union of subsets of I lies in $\mathcal{U}$ then at least one summand lies in $\mathcal{U}$. Since no set $f^{-1}(O_i)$ lies in $\mathcal{U}$, this claim produces a contradiction.

We prove the contrapositive of the claim just mentioned, namely that $V, W \notin \mathcal{U}$ implies $V \cup W \notin \mathcal{U}$. If $V, W \notin \mathcal{U}$, then $X \setminus V, X \setminus W \in \mathcal{U}$, hence $(X \setminus V) \cap (X \setminus W) = X \setminus (V \cup W) \in \mathcal{U}$. Therefore $V \cup W \notin \mathcal{U}$.

[(1), $\Leftarrow$] Choose a collection $\mathcal{C}$ of closed subsets of $\mathcal{X}$ with the FIP. Let $I = X$ and let $\mathcal{F}$ be the filter on I generated by $\mathcal{C}$. Let $\mathcal{U}$ be an ultrafilter extending $\mathcal{F}$. Let $f\colon I \to X$ be the identity function. Choose ℓ so that $\lim_{\mathcal{U}} f = \ell$. Let $\mathcal{N}_\ell$ be the set of open neighborhoods of ℓ. If $O \in \mathcal{N}_\ell$, then $O = f^{-1}(O) \in \mathcal{U}$. This yields $\mathcal{C} \cup \mathcal{N}_\ell \subseteq \mathcal{U}$, showing that $\mathcal{C} \cup \mathcal{N}_\ell$ has the FIP. This proves that ℓ belongs to every closed set in $\mathcal{C}$, hence that $\ell \in \bigcap \mathcal{C}$.

[(2), $\Rightarrow$] Assume $\mathcal{X}$ is Hausdorff and that $\lim_{\mathcal{U}} f = \ell$ and $\lim_{\mathcal{U}} f = m$ for appropriate $\mathcal{U}, f, \ell, m$ with $\ell \neq m$. Choose disjoint open neighborhoods O_ℓ and O_m of ℓ and m respectively. Then $f^{-1}(O_\ell)$ and $f^{-1}(O_m)$ are disjoint sets in $\mathcal{U}$, a contradiction.

[(2), $\Leftarrow$] Now suppose that $\mathcal{X}$ is not Hausdorff and that $\ell \neq m$ are elements of $\mathcal{X}$ that cannot be separated by disjoint open sets. If $\mathcal{N}_\ell$ and $\mathcal{N}_m$ are the open neighborhood systems of these two points, then $\mathcal{N}_\ell \cup \mathcal{N}_m$ has the FIP. Let $I = X$, let $\mathcal{U}$ be an ultrafilter on I containing $\mathcal{N}_\ell \cup \mathcal{N}_m$, and let $f\colon I \to X$ be the identity function. Both $\lim_{\mathcal{U}} f = \ell$ and $\lim_{\mathcal{U}} f = m$ hold, establishing the contrapositive of (2), $\Leftarrow$. □