κ-complete ultrafilters.

The image of the diagonal embedding into an ultrapower, $\Delta\colon A \to \prod_{\mathcal{U}} A$, is onto iff it is impossible to partition I into $\leq |A|$ measure zero cells. Indeed, if $f\colon I \to A$ and $[f]_{\theta_{\mathcal{U}}}$ is not in the image of the diagonal embedding, then $\mathrm{coim}(f) = \{f^{-1}(a) \mid a \in A\}$ is a partition of I into $\leq |A|$ measure zero cells. Conversely if $\mathcal{Z} = \{Z_j \mid j \in J\}$ is a partition of I into $\leq |A|$ measure zero cells, then there will be a function $f\colon I \to A$ with coimage $\mathcal{Z}$ and for this function $[f]_{\theta_{\mathcal{U}}}$ will not be in the image of the diagonal embedding.

The preceding remarks together with the upcoming theorem show that the surjectivity of the diagonal embedding is related to completeness properties of the ultrafilter.

Definition 1. Let I be an infinite set. An ultrafilter $\mathcal{U}$ on I is *κ-complete* if it is closed under $(<\kappa)$-fold intersection.

Any ultrafilter is ω-complete. An ultrafilter is κ-complete for all κ iff it is principal.

Theorem 2. *If $\mathcal{U}$ is an ultrafilter on I, then $\mathcal{U}$ is κ-complete iff I has no partition into fewer than κ-many cells of measure zero. Hence if $\mathcal{U}$ is κ-complete and $|A| < \kappa$, then the diagonal embedding $\Delta\colon A \to \prod_{\mathcal{U}} A$ is onto.*

Proof. [$\Rightarrow$, contrapositive] Assume that $\mathcal{Z} = \{Z_j \mid j \in J\}$ is a partition of I with $|J| < \kappa$ and each Z_j of measure zero (i.e., $Z_j \notin \mathcal{U}$ for all $j \in J$). Then $I \setminus Z_j \in \mathcal{U}$ for all $j \in J$ and $\bigcap_{j\in J}(I \setminus Z_j) = \emptyset \notin \mathcal{U}$, so $\mathcal{U}$ is not κ-complete.

[$\Leftarrow$] Assume that $\lambda < \kappa$ and $\mathcal{S} = \{U_\alpha \mid \alpha < \lambda\} \subseteq \mathcal{U}$. We want to prove that $\bigcap \mathcal{S} \in \mathcal{U}$.

Consider the function $f\colon I \to \lambda \cup \{\infty\}$ defined by $f(i) =$ least α such that $i \notin U_\alpha$ if such α exists and $f(i) = \infty$ otherwise. Since the coimage of f is a partition of I into at most λ-many cells, some cell must belong to $\mathcal{U}$. Suppose that cell is $f^{-1}(\alpha)$ for some $\alpha < \lambda$. Then U_α and $f^{-1}(\alpha)$ are disjoint elements of $\mathcal{U}$, which is impossible since $\mathcal{U}$ is a proper filter. Therefore the big cell must be $f^{-1}(\infty)$, which equals $\bigcap \mathcal{S}$. This shows that $\bigcap \mathcal{S} \in \mathcal{U}$. $\square$