Definition 1. Let I be an infinite set. An ultrafilter $\mathcal{U}$ on I is *regular* if there is a subset $E \subseteq \mathcal{U}$ such that

(i) $|E| = |I|$, and

(ii) each $i \in I$ is contained in only finitely many elements of E.

Item (ii) is equivalent to the statement that any infinite subset of E has empty intersection. Of course, any finite subset of E has intersection in $\mathcal{U}$, so is nonempty.

Theorem 2. *Any infinite set supports a regular ultrafilter.*

Proof. It suffices to show that on any infinite set I there is a set $E \subseteq \mathcal{P}(I)$ satisfying (i) $|E| = |I|$, and $(ii)'$ E has the FIP, but any infinite subset of E has empty intersection. For if (i) and $(ii)'$ hold, then the Ultrafilter Lemma allows us to find a (necessarily regular) ultrafilter extending E.

Start with a set J whose size is the desired size of the index set. Let I be the set of finite subsets of J. Then $|I| = |J|$, as needed. For each $j \in J$ define $\hat{\jmath} = \{i \in I \mid j \in i\}$, and then take $E = \{\hat{\jmath} \mid j \in J\}$. Clearly (i) $|E| = |J| = |I|$ holds, since the function $j \mapsto \hat{\jmath}$ is bijective.

To prove $(ii)'$, realize that if $E_0 \subseteq E$, then $E_0 = \{\hat{\jmath} \mid j \in J_0\}$ for some subset $J_0 \subseteq J$ satisfying $|J_0| = |E_0|$. Moreover, $\bigcap E_0 = \bigcap_{j \in J_0} \hat{\jmath} = \{i \in I \mid J_0 \subseteq i\}$. If E_0 is finite then this intersection is nonempty, while if it is infinite the intersection is empty. $\square$

Theorem 3. *(Frayne, Morel, Scott) If A is an infinite set and $\mathcal{U}$ is a regular ultrafilter on I, then $|\prod_{\mathcal{U}} A| = |A|^{|I|}$.*

Proof. Since $\prod_{\mathcal{U}} A$ is a quotient of A^I it follows that $|\prod_{\mathcal{U}} A| \leq |A|^{|I|}$.

For the reverse inequality, choose $E \subseteq \mathcal{U}$ witnessing regularity. Let B be the set of finite sequences of elements of A. Since A is infinite, we have $|B| = |A|$. To prove the desired inequality it will suffice to exhibit a 1-1 function $\alpha \colon A^E \to \prod_{\mathcal{U}} B = (B^I)/\theta_{\mathcal{U}}$. We will explain how to assign to any $f \colon E \to A$ a function $\hat{f} \colon I \to B$ such that $f \neq g$ implies $\hat{f} \not\equiv_{\mathcal{U}} \hat{g}$. Then we take α to be $f \mapsto [\hat{f}]_{\theta_{\mathcal{U}}}$.

Now we start explaining: Linearly order E with the relation $<$. Choose $f \in A^E$, and define $\hat{f} \in B^I$ as follows: if $i \in I$, let $(e_1, \ldots, e_m)$ be the finite set of all $e \in E$ such that $i \in e$ ordered according to the relation $<$. Define $\hat{f}(i) = (f(e_1), \ldots, f(e_m)) \in B$.

It remains to check that $f \neq g$ implies $\hat{f} \not\equiv_{\mathcal{U}} \hat{g}$. Suppose that $f(e) \neq g(e)$ for some $e \in E$. For any $i \in e$, the element e will occur in the sequence $(e_1, \ldots, e_m)$, say $e = e_k$. Now $\hat{f}(i) = (\ldots, f(e_k), \ldots) \neq (\ldots, g(e_k), \ldots) = \hat{g}(i)$. This shows that $\hat{f}(i) \neq \hat{g}(i)$ for all $i \in e$. Since $e \in \mathcal{U}$ this means $\hat{f} \not\equiv_{\mathcal{U}} \hat{g}$. $\square$