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2. "$\mathbf{A}$ realizes $\Sigma(\mathbf{x})$" is the assertion that $\mathbf{A}$ satisfies the $L_{\infty,\omega}$-sentence

$$(\exists \mathbf{x})\left(\bigwedge_{\varphi(\mathbf{x})\in\Sigma(\mathbf{x})} \varphi(\mathbf{x})\right).$$

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If T is a theory, then $S_n(T)$ is the closed subset of $\text{Spec}(L(x_1, \ldots, x_n))$ consisting of n-types of T. Again, we have continuous restrictions:

$$S_0(T) \leftarrow S_1(T) \leftleftarrows S_2(T) \Lleftarrow \cdots$$

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2. Any formula with parameters in $\mathbf{A}$ that has a 'solution' in $\mathbf{B}$ already has a solution in $\mathbf{A}$.
   (For every $\varphi(\mathbf{a}, y)$, if $\mathbf{B} \models (\exists y)\varphi(\mathbf{a}, y)$, then $\mathbf{A} \models (\exists y)\varphi(\mathbf{a}, y)$.)

Proof.
[(1)⇒(2)] $(\exists y)\varphi(\mathbf{x}, y) \in \mathrm{tp}_{\mathbf{B}}(\mathbf{a})$ iff $(\exists y)\varphi(\mathbf{x}, y) \in \mathrm{tp}_{\mathbf{A}}(\mathbf{a})$.

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1. Show that if $A \subseteq B$ are infinite sets, considered as structures in the language of equality, then $A \prec B$. Hence $A \prec B$ iff $A \subseteq B$ and $A \equiv B$.
2. Considering $\mathbb{Q}$ as an ordered set, show that $\mathbf{A} \prec \mathbb{Q}$ iff both $\mathbf{A} \leq \mathbb{Q}$ and $\mathbf{A} \equiv \mathbb{Q}$.
3. Show that if $U \leq V$ are infinite dimensional $\mathbb{F}_2$-spaces, then $U \prec V$. Hence $U \prec V$ iff $U \leq V$ and $U \equiv V$.

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If time, discuss the use of Skolemization to provide an alternative proof of the Downward LS Theorem.