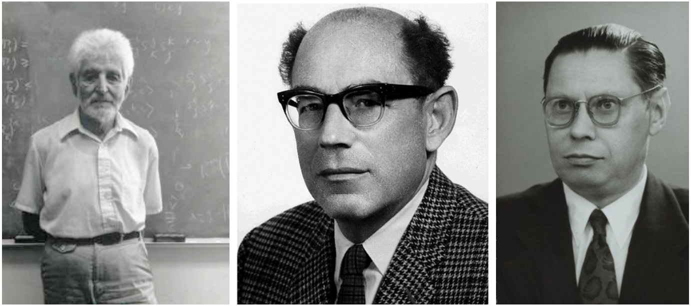
Three classical theorems



A general interpolation theorem

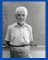
A basic “interpolation theorem” compares the strength of theories in different languages. It is usually proved by a refinement of the idea used for the Completeness Theorem and for the Omitting Types Theorem.

Basic Interpolation Theorem. Let T_1 be an L_1-theory and T_2 be an L_2-theory. If $T_1 \cup T_2$ is inconsistent, then there is a sentence θ in $L_1 \cap L_2$ that separates them. That is,

$$T_1 \models \theta \qquad \text{and} \qquad T_2 \models \neg\theta.$$

(You can model a proof on the proof of a similar result in Chang-Keisler, 2nd ed., pages 88-89.)

William Craig



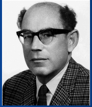
Craig Interpolation Theorem. If σ, τ are sentences and $\sigma \models \tau$, then there exists a sentence θ in the intersection language such that $\sigma \models \theta$ and $\theta \models \tau$.

Proof sketch. Let $T_1 = \{\sigma\}$ and $T_2 = \{\neg\tau\}$.
Since $\sigma \models \tau$, we get $T_1 \cup T_2 = \{\sigma, \neg\tau\} \models \tau \wedge \neg\tau$.
Hence $T_1 \cup T_2$ is inconsistent.
By the Basic Interpolation Theorem, there must be θ in the intersection language such that $\sigma \models \theta$ and $\neg\tau \models \neg\theta$.
The latter may be rewritten $\theta \models \tau$. $\square$

Example.

1. $\sigma = (\exists x)(P(x) \wedge \neg P(x))$.
2. $\tau = (\forall y)(Q(y) \vee \neg Q(y))$.
3. $\theta = (\exists y)(y = y)$.

Abraham Robinson



Robinson Joint Consistency Theorem. Let T_1 be a consistent L_1-theory and T_2 be a consistent L_2-theory. If $T_1 \cap T_2$ is a complete $L_1 \cap L_2$-theory, then $T_1 \cup T_2$ is consistent.

Proof sketch. If $T_1 \cup T_2$ is inconsistent, then Basic Interpolation guarantees that there exists an $L_1 \cap L_2$-sentence θ such that $T_1 \models \theta$ and $T_2 \models \neg\theta$.

Case 1. $T_1 \cap T_2 \models \theta$.
Then $T_2 \models \theta \wedge \neg\theta$, contradicting the consistency of T_2.

Case 2. $T_1 \cap T_2 \models \neg\theta$.
Then $T_1 \models \theta \wedge \neg\theta$, contradicting the consistency of T_1.

Since $T_1 \cap T_2$ is complete, these are the only cases. □

Evert Beth

Beth Definability Theorem. A relation is implicitly definable iff it is explicitly definable.

Example. Let $T = \mathrm{Th}(\langle \mathbb{R}; +, -, 0, \cdot, 1\rangle)$. Let $<$ be a new relation symbol. Implicitly define this relation by writing down the axioms of ordered fields.

1. $<$ is a strict linear order.
2. $(\forall a)(\forall b)(\forall c)((a < b) \to (a + c < b + c))$.
3. $(\forall a)(\forall b)(\forall c)(((a < b) \wedge (0 < c)) \to (ac < bc))$.

By the theorem this implicit definition is equivalent modulo T to an explicit definition, e.g.

$$(\forall a)(\forall b)((a < b) \leftrightarrow (\exists z)((z \neq 0) \wedge (a + z^2 = b))).$$

Implicit versus explicit definitions.

Implicit definition = use it in a sentence (or many sentences), until its meaning is uniquely determined.

Explicit definition = “dictionary definition”.

Example.

1. Colonel Mustard killed a Mammal in my Library with a Candlestick. The warm blood and bits of broken spinal column destroyed my comic book collection.
2. It is important for a Mammal to shave before a big job interview.
3. That Mammal was observed juggling three eggs, which obviously couldn’t be his.

versus

1. A **mammal** is: a warm-blooded invertebrate that possesses hair or fur that gives birth to live young.

Mathematical formulation.

Let L be a language, let R and R' be distinct n-relation symbols not in L. Let $\Sigma(R)$ be a set of $L \cup \{R\}$-sentences, and let $\Sigma(R')$ be the same set with R replaced by R'.

$\Sigma(R)$ defines R **implicitly** modulo an L-theory T iff

$$T \cup \Sigma(R) \cup \Sigma(R') \models (\forall \mathbf{x})(R(\mathbf{x}) \leftrightarrow R'(\mathbf{x})).$$

This means: the interpretation of R in any model of T is uniquely determined by $\Sigma(R)$.

An L-formula $\varphi(\mathbf{x})$ defines R **explicitly** modulo T if

$$T \models (\forall \mathbf{x})(R(\mathbf{x}) \leftrightarrow \varphi(\mathbf{x})).$$