The space of complete $L$-theories
We have shown that the space of complete $L$-theories is a compact, Hausdorff, zero-dimensional space.

**Theorem.** (Munkres, Thm 26.2) A closed subspace of a compact space is compact.

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**Urysohn's Metrization Theorem.** A normal space with a countable basis is metrizable.

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Example

Let $L$ be the language of one unary predicate symbol, $P(x)$. A typical structure in this language has the form $A = \langle A; P(x) \rangle$. The isomorphism type of $A$ is determined by the pair of cardinals $(\kappa, \lambda)$ where $|P[A]| = \kappa$ and $|A - P[A]| = \lambda$. Since every complete $L$-theory has a countable model, the complete theory of $A$ is determined by the 'nearest' pair $(\kappa, \lambda)$ for which $\kappa, \lambda \in \{0, 1, 2, ..., \omega\}$. The space of complete $L$-theories looks like:

\[
\begin{array}{ccccccc}
\text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} \\
\text{...} & \text{isolated point} & \text{...} & \text{isolated point} & \text{...} & \text{isolated point} & \text{...} \\
\text{...} & \text{nonisolated point} & \text{...} & \text{nonisolated point} & \text{...} & \text{nonisolated point} & \text{...} \\
\text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} & \text{...} \\
\end{array}
\]

The space of complete $L$-theories 3/6
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```
. . . . . .
. . . . . .
. . . . . .
. . . . . .
. . . . . .
... ... ...
   ···   ···
   ···   ···
   ···   ···
   ···   ···
   ···   ···
... ... ...
```

→ isolated point
← nonisolated point

= a finitely axiomatizable theory
= a nonfinitely axiomatizable theory
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```
  . . . . . . . . .
  . . . . . . . . .
  . . . . . . . . .
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  . . . . . . . . .
  . . . . . . . . .
  . . . . . . . . .
  . . . . . . . . .
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- isolated point $\rightarrow$
- nonisolated point $\leftarrow$

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```
. . . . . . . . .
::: : : : : : : :
. . . . . . . . .
. . . . . . . . . ← nonisolated point
isolated point $\rightarrow$
. . . . . . . . .
::: : : : : : : :
. . . . . . . . .
. . . . . . . . .
```

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  . . . . . . . .
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\[\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\circ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}\]

isolated point $\rightarrow$ = a finitely axiomatizable theory

$\leftarrow$ nonisolated point = a nonfinitely axiomatizable theory
Theorem. Let $\mathcal{L}$ be a countable language. If $X = V(\mathcal{T})$ is a closed subspace of the space of complete $\mathcal{L}$-theories, then $X$ is uniquely representable as the disjoint union $S \cup P$ where $S$ is a countable, open, scattered subspace and $P$ is a possibly empty perfect subspace. If $P \neq \emptyset$, then $P$ is homeomorphic to the Cantor space. Moreover, each complete theory $\mathcal{T}'$ in the scattered part of $X = V(\mathcal{T})$ may be assigned an ordinal rank, its Cantor-Bendixson rank, which may be interpreted as a measure of how difficult it is to axiomatize $\mathcal{T}'$ relative to $\mathcal{T}$.

Some Definitions. A point $p$ of a topological space $X$ is isolated if $\{p\}$ is an open set in $X$. Otherwise $p$ is nonisolated (or a limit point of $X$). A space $X$ is scattered if every nonempty subspace $V \subseteq X$ contains a point that is isolated in $V$. A closed subspace $P \subseteq X$ is perfect if it contains no point that is isolated in $P$. The space of complete $\mathcal{L}$-theories
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**Some Definitions.**
Application of TOP to the space of complete theories

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Cantor-Bendixson Rank

Definition. The Cantor-Bendixson derivative of a space $X$ is the subspace $X'$ consisting of the nonisolated points. Recursively define $X_0 := X$. If $X_\alpha$ is the Cantor-Bendixson derivative, then

$$X_{\alpha+1} := X_\alpha'$$

and

$$X_\lambda := \bigcap_{\alpha < \lambda} X_\alpha$$

if $\lambda$ is limit.

The Cantor-Bendixson rank of $X$ is the least $\alpha$ such that $X_\alpha = X_{\alpha+1}$.

I will say that point $p \in X$ has Cantor-Bendixson rank $\alpha$ if $p \in X_\alpha - X_{\alpha+1}$.

If $p$ is not assigned an ordinal rank, then say that $p$ has Cantor-Bendixson rank $\infty$.

The structure of the proof of the Cantor-Bendixson Theorem is to show that a space $X$ satisfying the hypotheses of the theorem decomposes (uniquely) as $X = S \cup P$ where $S$ is the subspace of elements with rank $< \infty$, and $P$ is the subspace of elements with rank $= \infty$.

The space of complete $L$-theories
Cantor-Bendixson Rank

Definition.
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Recursively define

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- $X_{\alpha+1} := X'_\alpha$
- $X_\lambda := \bigcap_{\alpha < \lambda} X_\alpha$, if $\lambda$ is a limit.
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\end{align*}$$

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The space of complete $L$-theories
Cantor-Bendixson Rank

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The Cantor-Bendixson rank of $X$ is the least $\alpha$ such that $X_\alpha = X_{\alpha+1}$.

*I* will say that point $p \in X$ has Cantor-Bendixson rank $\alpha$ if $p \in X_\alpha - X_{\alpha+1}$. If $p$ is not assigned an ordinal rank, then say that $p$ has Cantor-Bendixson rank $\infty$.

The structure of the proof of the Cantor-Bendixson Theorem is to show that a space $X$ satisfying the hypotheses of the theorem decomposes (uniquely) as $X = S \cup P$ where $S$ is the subspace of elements with rank $< \infty$, and $P$ is the subspace of elements with rank $= \infty$. 
Proof of the Cantor-Bendixson Theorem

Theorem. Any closed subspace of a metric space with a countable basis is uniquely representable as the disjoint union of a countable, open, scattered subspace and a possibly empty perfect subspace.

Proof. (Existence of decomposition only.)

The sequence $X = X_0 \supseteq X_1 \supseteq \cdots$ of derived subspaces is a descending sequence of closed subspaces of $X$, and each $X_\alpha$ satisfies all the assumptions we made about $X$.

Let $P = \bigcap X_\alpha =$ set of points of rank $\infty$, and let $S = X - P =$ set of points of rank $< \infty$. $P$ is closed and $S$ is open.

For each point $p \in S$ there is some $\alpha < \infty$ such that $p \in X_\alpha - X_{\alpha+1}$.

The point $p$ must be isolated in $X_\alpha$, so $X$ has a basic open set $B_p$ such that $B_p \cap X_\alpha = \{p\}$.

The map $p \mapsto B_p$ is an injective function from $S$ to the countable basis of $X$. This shows that $S$ is countable.

Each nonempty subset $V \subseteq S$ has a point of least rank, and this point must be isolated in $V$, showing that $S$ is scattered.

$P = X_\alpha$ for some (the least) $\alpha$ such that $X_\alpha = X_{\alpha+1} = X'$.

This implies that $P$ is perfect.

The space of complete $L$-theories
Proof of the Cantor-Bendixson Theorem

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Let $P = \bigcap X_\alpha = \text{set of points of rank } \infty$, and let $S = X - P = \text{set of points of rank } < \infty$. $P$ is closed and $S$ is open.

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