The space of complete L-theories




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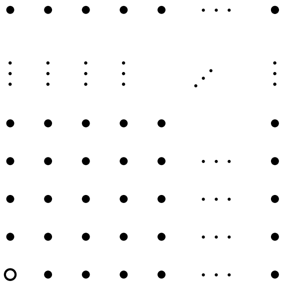

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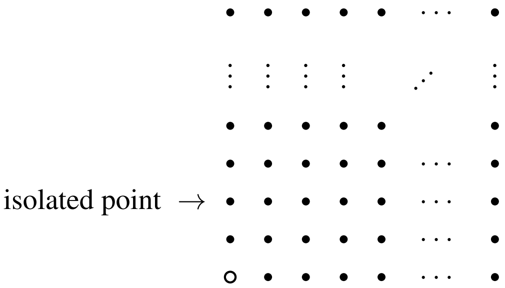

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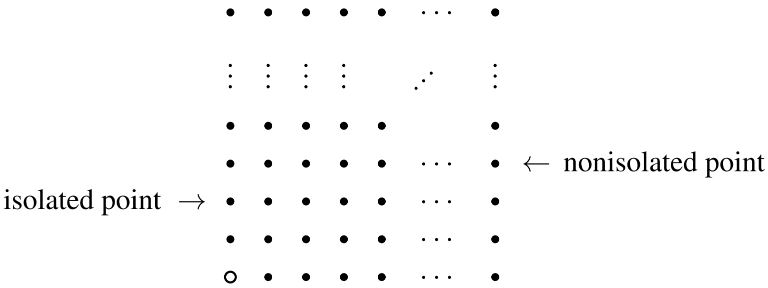

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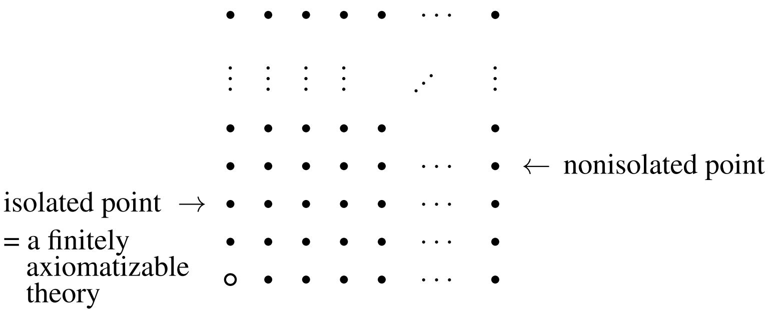

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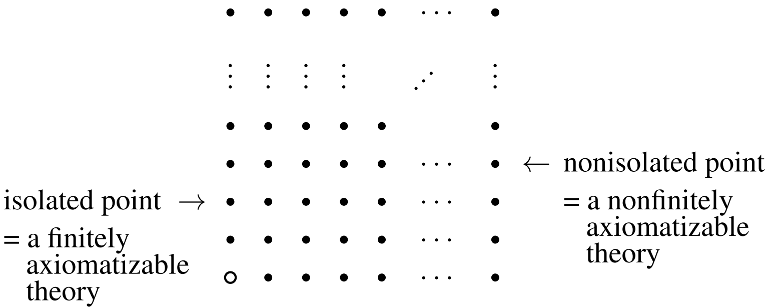

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The structure of the proof of the Cantor-Bendixson Theorem is to show that a space X satisfying the hypotheses of the theorem decomposes (uniquely) as $X = S \cup P$ where S is the subspace of elements with rank $< \infty$, and P is the subspace of elements with rank $= \infty$.

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The sequence $X = X_0 \supseteq X_1 \supseteq \cdots$ of derived subspaces is a descending sequence of closed subspaces of X, and each X_α satisfies all the assumptions we made about X. Let $P = \bigcap X_\alpha =$ set of points of rank ∞, and let $S = X - P =$ set of points of rank $< \infty$. P is closed and S is open.

For each point $p \in S$ there is some $\alpha < \infty$ such that $p \in X_\alpha - X_{\alpha+1}$. The point p must be isolated in X_α, so X has a basic open set B_p such that $B_p \cap X_\alpha = \{p\}$. The map $p \mapsto B_p$ is an injective function from S to the countable basis of X. This shows that S is countable.

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