# Saturated models

# Models realizing many types

Throughout these slides, $T$ will be a complete theory in a countable language which has infinite models.

By the Compactness Theorem, any model of $T$ has an elementary extension that realizes all types.

One expects such an extension to behave like a "completion" or "compactification" of the original model.

**Defn.** Call a model $\mathbf{S}$ of $T$ *weakly saturated* if it realizes all types in $S_n(T)$ for all $n$.

# Models realizing many types

The definition of "weakly saturated model" seems dual to the definition of atomic model, so in an ideal world, the following would be true:

1. Countable weakly saturated models of $T$ would exist.
2. Any two would be isomorphic.
3. Any countable model of $T$ would embed elementarily into the weakly saturated model.
4. Two tuples in a weakly saturated model would have the same type iff they differed by an automorphism.

But all of these statements are false.
The first statement becomes true provided $|S_n(T)| < 2^{\aleph_0}$ for all $n$. And then all statements become true with "$\omega$-saturated" in place of "weakly saturated".

# The Ehrenfeucht theory

**Example.** Let $T$ be the theory of dense linear order without endpoints expanded by a strictly increasing $\omega$-chain of constants.

1. Signature involves $<, c_0, c_1, \ldots$ only.
2. Axioms for $T$ =
   - (i) Axioms of dense linear orders without endpoints.
   - (ii) $c_i < c_{i+1}$ for each $i$.
3. Theory has q.e. and is complete.
4. The theory has three isomorphism types of countable models. ($I(T, \omega) = 3$.) Any countable model is isomorphic to one of the form $\langle \mathbf{Q}; <, c_0, c_1, \ldots \rangle$ where
   - (i) (Model $\mathbf{M}_1$) The sequence $(c_i)_{i\in\omega}$ is unbounded.
   - (ii) (Model $\mathbf{M}_2$) The sequence $(c_i)_{i\in\omega}$ has a least upper bound in the model.
   - (iii) (Model $\mathbf{M}_3$) The sequence $(c_i)_{i\in\omega}$ has an upper bound in the model but has no least upper bound in the model.

# The countable models $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3$

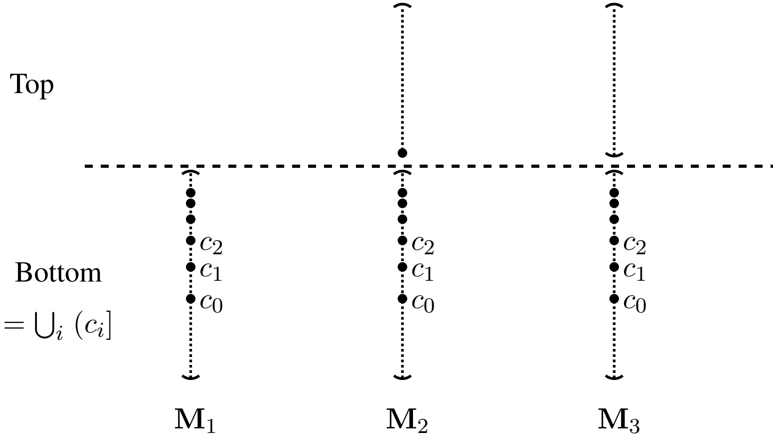


Figure: Top$(x) = \{c_0 < x, \;\; c_1 < x, \;\; c_2 < x, \;\; \ldots\}$, nonisolated $p \in S_1(T)$

# Observations

1. The fact that $I(T, \omega) = 3$ can be checked by noting that a countable model is determined up to isomorphism by the part above all the constants, and that part is a (possibly empty) dense linear order without top element.
2. All embeddings between models are elementary by q.e. $\mathbf{M}_1 \prec \mathbf{M}_2 \prec \mathbf{M}_3 \prec \mathbf{M}_2$.
3. $I(T, \omega) < 2^{\aleph_0}$ implies $S_n(T)$ is scattered for all $n$, so one of the models must be atomic. The only plausible candidate is $\mathbf{M}_1$.
4. All countable models embed elementarily into both $\mathbf{M}_2$ and $\mathbf{M}_3$. This is enough to prove that $\mathbf{M}_2$ and $\mathbf{M}_3$ are both weakly saturated.
5. The model $\mathbf{M}_2$ does not have the type-extension property. Let $p \in S_1(T)$ be the type $p(x_1) = \mathrm{Top}(x_1)$. Let $q \in S_2(T)$ be the type $q(x_1, x_2) = \mathrm{Top}(x_1) \cup \mathrm{Top}(x_2) \cup \{x_2 < x_1\}$. $q|_1 = p$. Let $a = \mathrm{lub}(c_i)$. The 1-tuple $(a)$ realizes $p$. Some 1-tuples that realize $p$ can be extended to 2-tuples that realize $q$. But the 1-tuple $(a)$ cannot be extended to a 2-tuple that realizes $q$.

# Tweaking the example by coloring the points

We introduce two new unary relations, red$(x)$ and blue$(x)$. Our goal is to construct a theory like Ehrenfeucht's, but with every point colored either red or blue, but not both colors.

1. Signature involves $<$, red$(x)$, blue$(x)$, $c_0, c_1, \ldots$ only.
2. Axioms for $T$ =
   (i) Axioms of dense linear orders without endpoints.
   (ii) An axiom saying that each point has a unique color:

$$(\forall x)((\text{red}(x) \wedge \neg\text{blue}(x)) \vee (\neg\text{red}(x) \wedge \text{blue}(x))).$$

   (iii) Both red points and blue points are dense:

$$(\forall w)(\forall x)((w < x) \rightarrow (\exists y)(\exists z)(\text{red}(y) \wedge \text{blue}(z) \wedge (w < y < x) \wedge (w < z < x))).$$

   (iv) $c_i < c_{i+1}$ for each $i$.
   (v) red$(c_i)$ for each $i$.

# This theory has four countable models

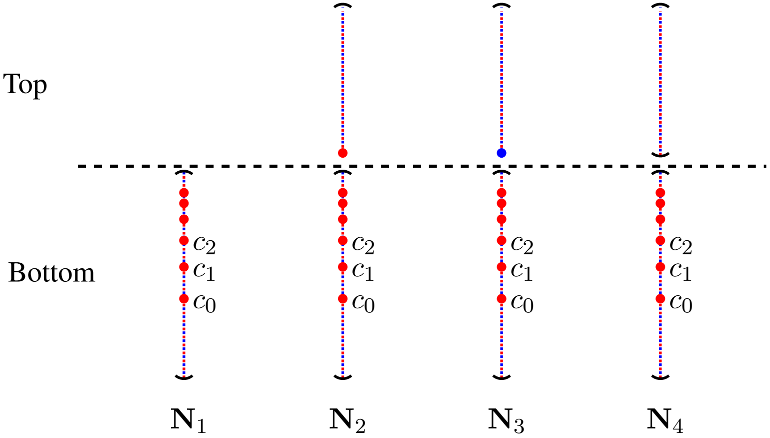


Figure: $\mathbf{N}_1 \prec \mathbf{N}_2 \prec \mathbf{N}_3 \prec \mathbf{N}_4 \prec \mathbf{N}_2$

# More observations about the uncolored version

1. $\mathbf{M}_2$ (the model where $\text{lub}(c_i)$ exists in the model) does not have the type-extension property. The problem involves the bound $a = \text{lub}(c_i)$.
2. If $a = \text{lub}(c_i)$, then $p = \{c_i < x < a \mid i \in \omega\}$ is a 1-type in $L_a$, which is not realized in $(\mathbf{M}_2)_a$. Thus, $\mathbf{M}_2$ is weakly saturated, while an expansion by a single constant is no longer weakly saturated.
3. All upper bounds of the sequence $(c_i)_{i \in \omega}$ have the same 1-type over the empty set (namely $\text{Top}(x)$). But $a = \text{lub}(c_i)$ does not differ from other realizations of $\text{Top}(x)$ by an automorphism.
4. On the other hand, $\mathbf{M}_3$ does have the type-extension property, any expansion of $\mathbf{M}_3$ by finitely many constants is again weakly saturated, and any two tuples of the same type in $\mathbf{M}_3$ differ by an automorphism. $\mathbf{M}_3$ is $\omega$-saturated.

# Isomorphism

Let $\mathbf{A}$ and $\mathbf{B}$ be countable structures both enumerated by $\omega$:
$\mathbf{A} = \{a_0, a_1, a_2, \ldots\}$
$\mathbf{B} = \{b_0, b_1, b_2, \ldots\}$
The assignment $a_i \mapsto b_i$ is an isomorphism iff it is type-preserving:

$$\text{tp}(a_0, \ldots, a_{n-1}) = \text{tp}(b_0, \ldots, b_{n-1}) \tag{1}$$

for all $n$.

Suppose we want to build an isomorphism one element at a time, by ensuring that, given equality of types of length-$n$ initial segments $\mathbf{a}$, $\mathbf{b}$, as in (1), and given the choice for $a_n$, we can find a corresponding choice for $b_n$. If we work only with types over the empty set, then we need some form of the type-extension lemma. It is enough to assume $\mathbf{A}$ and $\mathbf{B}$ are weakly saturated PLUS any two tuples of the same type differ by an automorphism. OR, we can work with $\mathbf{A}_{\mathbf{a}}$ and $\mathbf{B}_{\mathbf{b}}$ and then deal only with types in the expanded language $L_{\mathbf{a}}$.

**Defn.** Let $T$ be a complete theory.

1. A model $\mathbf{M}$ of $T$ is *$\omega$-saturated* if, whenever $\mathbf{a} \in M^n$, $\mathbf{M_a}$ realizes every type in $S_1(\mathbf{a})$. Often written "whenever $A \subseteq \mathbf{M}$, $|A| < \omega$, $\mathbf{M}_A$ realizes every type in $S_1(A)$". (Equivalently, $\mathbf{M}_A$ realizes every type in $S_n(A)$ for each finite $n$, Proposition 4.3.2, Marker.)
2. (Type extension) A model $\mathbf{M}$ of $T$ is *$\omega$-homogeneous* if, whenever $\mathbf{a}, \mathbf{b} \in M^n$, $\mathrm{tp}(\mathbf{a}) = \mathrm{tp}(\mathbf{b})$, and $c \in M$, then there exists $d \in M$ such that $\mathrm{tp}(\mathbf{a}c) = \mathrm{tp}(\mathbf{b}d)$.
3. A model $\mathbf{M}$ of $T$ is *strongly $\omega$-homogeneous* if, whenever $\mathbf{a}, \mathbf{b} \in M^n$, $\mathrm{tp}(\mathbf{a}) = \mathrm{tp}(\mathbf{b})$, then there is an automorphism $\alpha$ of $\mathbf{M}$ such that $\alpha(\mathbf{a}) = \mathbf{b}$.
4. A model $\mathbf{M}$ of $T$ is *$\omega^+$-universal* if every countable model of $T$ is elementarily embeddable in $\mathbf{M}$.

# Relationships

**Theorem.** Let $T$ be a complete theory in a countable language. TFAE about a countable model $\mathbf{M}$ of $T$.

1. $\mathbf{M}$ is $\omega$-saturated.
2. $\mathbf{M}$ is weakly saturated and $\omega$-homogeneous. ($\mathbf{M}$ realizes all types over the empty set and has the type extension property.)
3. $\mathbf{M}$ is weakly saturated and strongly $\omega$-homogeneous.
4. $\mathbf{M}$ is $\omega^+$-universal and $\omega$-homogeneous.
5. $\mathbf{M}$ is $\omega^+$-universal and strongly $\omega$-homogeneous.

**Trivial implications.**
$\omega^+$-universality implies weak saturation.
Strong $\omega$-homogeneity implies $\omega$-homogeneity.

**Not-too-hard implications.**
$\omega$-saturation implies strong $\omega$-homogeneity. (Back and forth.)
$\omega$-saturation implies $\omega^+$-universality. (Forth.)

# Weak saturation and $\omega$-homogeneity in E's Theory

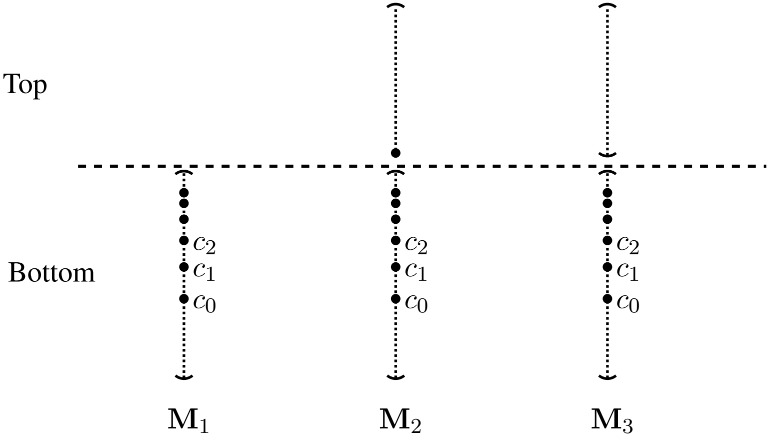


Figure: $\mathbf{M}_2, \mathbf{M}_3$ weakly saturated; $\mathbf{M}_1, \mathbf{M}_3$ $\omega$-homogeneous

# A proof sketch

**Theorem.** Two countable $\omega$-saturated models of $T$ are isomorphic. (Back and forth.)

Assume $\mathbf{A}$ and $\mathbf{B}$ are $\omega$-saturated models of $T$.

Enumerate them.

Start back and forth: Assume that $f : \mathbf{a} \to \mathbf{b}$ is a partial isomorphism that we want to extend. At this point, $\mathrm{tp}^{\mathbf{A}}(\mathbf{a}) = \mathrm{tp}^{\mathbf{B}}(\mathbf{b})$. Equivalently, $\mathbf{A} \models \varphi(\mathbf{a})$ iff $\mathbf{B} \models \varphi(\mathbf{b})$. Equivalently, $\mathbf{A}_{\mathbf{a}} \equiv \mathbf{B}_{\mathbf{b}}$.

Assume it is our turn to extend the domain. Let $c \in \mathbf{A}$ be the least unconsidered element. Let $p = \mathrm{tp}^{\mathbf{A}_{\mathbf{a}}}(c)$. Let $d$ be a realization of $p$ in $\mathbf{B}_{\mathbf{b}}$. Thus, $\mathbf{A} \models \theta(\mathbf{a}c)$ iff $\mathbf{B} \models \theta(\mathbf{b}d)$. I.e., $\mathrm{tp}^{\mathbf{A}}(\mathbf{a}c) = \mathrm{tp}^{\mathbf{B}}(\mathbf{b}d)$. Extend $f$ so that $f(c) = d$. $\square$

When $\mathbf{A} = \mathbf{B}$, this argument proves strong $\omega$-homogeneity of $\omega$-saturated models. Half of the argument proves $\omega^+$-universality.

**Theorem.** Let $T$ be a complete theory in a countable language. If $\mathbf{A}$ and $\mathbf{B}$ are countable $\omega$-saturated models of $T$, then $\mathbf{A} \cong \mathbf{B}$.

**Theorem.** Let $T$ be a complete theory in a countable language. TFAE.

1. $T$ has a countable $\omega$-saturated model.
2. $T$ has a countable weakly saturated model.
3. $T$ is "small". ($|S_n(T)| < 2^{\aleph_0}$ for all $n$.)

*Part of proof.*
$(1) \Rightarrow (2) \Rightarrow (3)$ uses only ideas we have seen.
$(3)$ implies that, for any model $\mathbf{M}$ of $T$, and any finite subset $A \subseteq \mathbf{M}$, ($|A| = m$, say), then $|S_n^{\mathbf{M}}(A)| \leq |S_{m+n}(T)| \leq \omega$.
*Idea for the rest.* Let $\mathbf{M}_0 = \mathbf{M}$. Find a countable elementary extension $\mathbf{M}_{i+1} \succ \mathbf{M}_i$ that realizes the countable set of 1-types over finite subsets of $\mathbf{M}_i$. Let $\widehat{\mathbf{M}}$ be the union of the $\mathbf{M}_i$. $\square$

# Extensions to higher cardinalities

**Defn.** A model $\mathbf{M}$ of $T$ is *$\kappa$-saturated* if whenever $A \subseteq M$ satisfies $|A| < \kappa$, then $\mathbf{M}_A$ realizes all $p \in S_1^{\mathbf{M}}(A)$. We say that $\mathbf{M}$ is *saturated* if it is $|\mathbf{M}|$*-saturated*.

To discuss this when $\kappa \neq \omega$, we need a concept of type for infinitely long tuples.

**Some basic results.**

1. $\kappa$-saturated = $\kappa^+$-universal and $\kappa$-homogeneous.
2. Formation of ultrapowers increases saturation.
3. An infinite model $\mathbf{M}$ satisfying $|\mathbf{M}| \leq 2^\kappa$ has a $\kappa^+$-saturated elementary extension of cardinality $2^\kappa$.

# Saturated models of $\mathrm{ACF}_p$

**Theorem.** A model $\mathbf{K}$ of $\mathrm{ACF}_p$ is saturated if(f) it contains an algebraically independent subset of size $|\mathbf{K}|$.

*Proof sketch.* Let's explain why $\mathbb{C}$ is saturated.
Let $A \subseteq \mathbb{C}$ satisfy $|A| < |\mathbb{C}|$.
Let $\mathbf{F}$ be an algebraically closed subfield of $\mathbb{C}$ containing $A$ and satisfying $|\mathbf{F}| < |\mathbb{C}|$.
By q.e., types over $A$ are determined by $\pm$atomic part.
Any type over $A$ with a "+atomic" part ($p(x, \mathbf{a}) = 0$) is realized in $\mathbf{F}$, hence in $\mathbb{C}$.
Any complete type over $A$ with only "$-$atomic" describes an element transcendental over $\mathbf{F}$. $\mathbb{C}$ has such an element. $\Box$.

# Appendix: Ultrapowers are somewhat saturated

Let $\mathbf{A}$ be a structure and $B \subseteq \mathbf{A}$ be a subset. Let $\kappa = \|L_B\|$. Let $I = \mathcal{P}_{\text{fin}}(\kappa)$ be the set of finite subsets of $\kappa$.

$I = \mathcal{P}_{\text{fin}}(\kappa)$ is directed by inclusion. The tail ends of this directed set form a proper filter on $I$, which can be extended to an ultrafilter $\mathcal{U}$ on $I$. Let's outline why $\prod_{\mathcal{U}} \mathbf{A}$ realizes every type in $S_1^{\mathbf{A}}(B)$.

Accept for now that every $p \in S_1^{\mathbf{A}}(B)$ has cardinality $\kappa = \|L_B\|$, and choose a bijection $\beta_p : \kappa \to p$. There is an induced bijection from $I = \mathcal{P}_{\text{fin}}(\kappa)$ to $\mathcal{P}_{\text{fin}}(p)$, which we also call $\beta_p$. Thus, for each $i \in I$, there is assigned a set $\beta_p(i)$, which is a finite subset of $p$.

Since $p$ is consistent with $\text{Th}(\mathbf{A}_B)$, for each $i$ there is an element $a_i \in \mathbf{A}$ that satisfies all formulas in the finite set $\beta_p(i)$. Let $\mathbf{a} \in \mathbf{A}^I$ be the tuple satisfying $(\mathbf{a})_i = a_i$ for all $i$. For each $\varphi(x) \in p$ we have that $[\![\varphi[\mathbf{a}]]\!]$ contains the tail end in $I = \mathcal{P}_{\text{fin}}(\kappa)$ generated by $\beta_p^{-1}(\varphi(x))$. This tail end belongs to $\mathcal{U}$, hence $\prod_{\mathcal{U}} \mathbf{A} \models \varphi[\mathbf{a}]$. This is true for any $\varphi(x) \in p$, so $\mathbf{a}$ realizes $p$ in $\prod_{\mathcal{U}} \mathbf{A}$.

Similarly, every $q \in S_1^{\mathbf{A}}(B)$ is realized in the same ultrapower. It is worth recording that $|\prod_{\mathcal{U}} \mathbf{A}| = |\mathbf{A}|^{\|L_B\|}$.

# Appendix to the appendix

On the previous slide it was claimed that all types in $S_1^{\mathbf{A}}(B)$ have the same cardinality, namely $\kappa = \|L_B\|$. This fact was used so that we could correspond finite subsets of any $p \in S_1^{\mathbf{A}}(B)$ to finite subsets of the fixed set $\kappa$. This was needed so that a fixed ultrapower was able to realize all types in $S_1^{\mathbf{A}}(B)$ simultaneously.

One can prove that all types in $S_1^{\mathbf{A}}(B)$ have size $\kappa = \|L_B\|$ as follows.

1. If $p \in S_1^{\mathbf{A}}(B)$, then $p \subseteq L_B$, so $|p| \leq \|L_B\| = \kappa$.
2. The existence of the map $\varphi(\mathbf{x}) \mapsto (\forall x_1) \cdots (\forall x_k)\varphi(\mathbf{x})$, which assigns to a formula its universal closure, is a finite-to-one map from $L_B$ to a subset of the $L_B$-sentences. This establishes that the set of $L_B$-sentences has size at least $\kappa$. (I am basing this claim on the fact that if $X$ and $Y$ are infinite and there is a finite-to-one map from $X$ into $Y$, then $|X| \leq |Y|$.)
3. Any $p \in S_1^{\mathbf{A}}(B)$ contains half of the $L_B$-sentences, hence $p$ has cardinality at least $\frac{1}{2}\kappa = \kappa$.