Saturated models

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Defn. Call a model $\mathbf{S}$ of T *weakly saturated* if it realizes all types in $S_n(T)$ for all n.

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The first statement becomes true provided $|S_n(T)| < 2^{\aleph_0}$ for all n. And then all statements become true with "ω-saturated" in place of "weakly saturated".

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The countable models $\mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3$

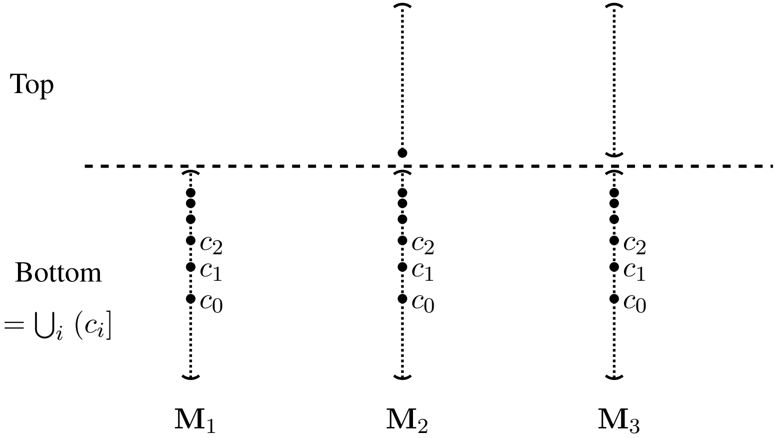

Figure: Top$(x) = \{c_0 < x, \;\; c_1 < x, \;\; c_2 < x, \;\; \ldots\}$, nonisolated $p \in S_1(T)$

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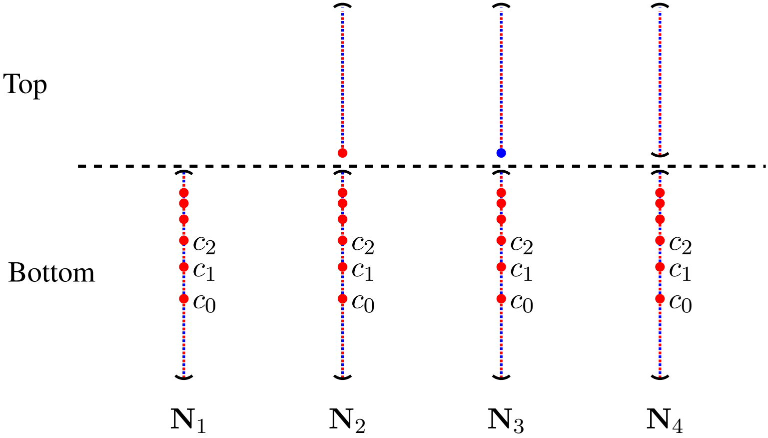

Figure: $\mathbf{N}_1 \prec \mathbf{N}_2 \prec \mathbf{N}_3 \prec \mathbf{N}_4 \prec \mathbf{N}_2$

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Suppose we want to build an isomorphism one element at a time, by ensuring that, given equality of types of length-n initial segments $\mathbf{a}$, $\mathbf{b}$, as in (1), and given the choice for a_n, we can find a corresponding choice for b_n.

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The assignment $a_i \mapsto b_i$ is an isomorphism iff it is type-preserving:

$$\mathrm{tp}(a_0, \ldots, a_{n-1}) = \mathrm{tp}(b_0, \ldots, b_{n-1}) \tag{1}$$

for all n.

Suppose we want to build an isomorphism one element at a time, by ensuring that, given equality of types of length-n initial segments $\mathbf{a}$, $\mathbf{b}$, as in (1), and given the choice for a_n, we can find a corresponding choice for b_n. If we work only with types over the empty set, then we need some form of the type-extension lemma.

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Weak saturation and ω-homogeneity in E's Theory

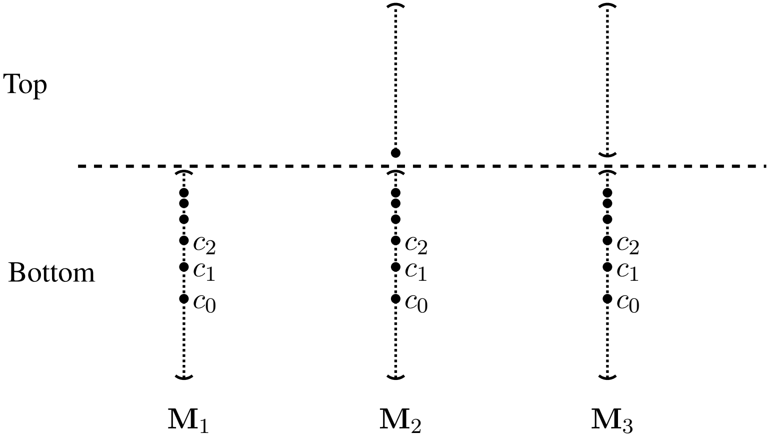

Figure: $\mathbf{M}_2, \mathbf{M}_3$ weakly saturated; $\mathbf{M}_1, \mathbf{M}_3$ ω-homogeneous

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1. If $p \in S_1^{\mathbf{A}}(B)$, then $p \subseteq L_B$, so $|p| \leq \|L_B\| = \kappa$.
2. The existence of the map $\varphi(\mathbf{x}) \mapsto (\forall x_1) \cdots (\forall x_k)\varphi(\mathbf{x})$, which assigns to a formula its universal closure, is a finite-to-one map from L_B to a subset of the L_B-sentences.

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One can prove that all types in $S_1^{\mathbf{A}}(B)$ have size $\kappa = \|L_B\|$ as follows.

1. If $p \in S_1^{\mathbf{A}}(B)$, then $p \subseteq L_B$, so $|p| \leq \|L_B\| = \kappa$.
2. The existence of the map $\varphi(\mathbf{x}) \mapsto (\forall x_1) \cdots (\forall x_k)\varphi(\mathbf{x})$, which assigns to a formula its universal closure, is a finite-to-one map from L_B to a subset of the L_B-sentences.

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