Roundup: Ultraproducts

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Thus, ultrapowers may be used to construct arbitrarily large elementary extensions of infinite structures.

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Characterizing the closure of a class of L-structures

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4. (Keisler-Shelah) $\mathcal{J} = \sqrt[u]{\mathsf{P}_u(\mathcal{K})}$. ($\sqrt[u]{}$ = closure under 'ultraroots'.)