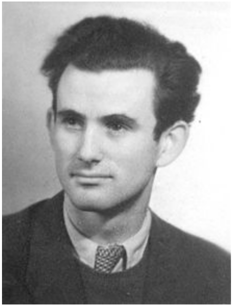
Universal classes



The class of models of a set of universal sentences

We are going to discuss the relationship between the "syntactic form" of the sentences in a set Δ of first-order sentences and the class of models of Δ.

It is not hard to show that if Δ is a set of universal sentences, then the class of models of Δ is closed under the formation of substructures.

In general, the converse is false: The class of finite sets is closed under the formation of substructures, but it is not axiomatizable by any set of first-order sentences.

However, it is true that if $\mathbb{K}$ is a first-order axiomatizable class of structures, then $\mathbb{K}$ is closed under the formation of substructures if and only if $\mathbb{K}$ has a set of axioms that are universal sentences.

Beware that $((\forall x)(2x = 0) \to (\forall x)(x = 0))$ is not a "universal sentence" in the language of abelian groups, although it is a sentence whose only quantifiers are universal.

A technical lemma

Lemma. Let T be a satisfiable L-theory. Let Δ be a set of L-sentences that is closed under disjunction. TFAE.

1. Δ contains a set of axioms for T,
2. For any L-structures $\mathbb{A}$, $\mathbb{B}$, if
   (a) $\mathbb{A} \models T$, and
   (b) $(\forall \sigma \in \Delta)((\mathbb{A} \models \sigma) \quad \text{implies} \quad (\mathbb{B} \models \sigma))$,

   then $\mathbb{B} \models T$.

Proof.
$(1) \Rightarrow (2)$: Let $\Delta_0 \subseteq \Delta$ be a set of axioms for T. Choose L-structures $\mathbb{A}, \mathbb{B}$. Assume $(2)(a)$, $\mathbb{A} \models T$. Apply $(2)(b)$ to each $\sigma \in \Delta_0$. You obtain $\mathbb{B} \models \Delta_0$, so $\mathbb{B} \models T$.

$(2) \Rightarrow (1)$

$(2) \Rightarrow (1)$: Assume (2) and let Δ_0 be the set of all $\sigma \in \Delta$ such that $T \models \sigma$. We must show that $\Delta_0 \models T$.

Claim. If $\mathbb{B} \models \Delta_0$, then $\mathbb{B} \models T$.

Let $\Gamma = \{\neg\gamma \mid \mathbb{B} \models \neg\gamma,\ \gamma \in \Delta\}$.

Subclaim. $T \cup \Gamma$ is consistent.
Else there exist $\neg\gamma_1, \ldots, \neg\gamma_n \in \Gamma$ such that $T \models \neg(\neg\gamma_1 \wedge \cdots \wedge \neg\gamma_n)$. Hence $\gamma_1 \vee \cdots \vee \gamma_n \in \Delta$ is a consequence of T. Hence $\gamma_1 \vee \cdots \vee \gamma_n \in \Delta_0$. Hence $\mathbb{B} \models \gamma_1 \vee \cdots \vee \gamma_n$. This contradicts $\mathbb{B} \models \neg\gamma_1, \ldots, \mathbb{B} \models \neg\gamma_n$. $\Box$ (Subclaim)

Let $\mathbb{A}$ be a model of $T \cup \Gamma$. Apply $(2)(b)$ to conclude that $\mathbb{B} \models T$. $\Box$ (Claim and Theorem)

Theories axiomatizable by universal sentences

Łos-Tarski Theorem.
The class of models of T is closed under the formation of substructures if and only if T has a set of axioms that are universal sentences. (I.e., Π_1-sentences.)

Proof. ($\Leftarrow$)
Assume that T is axiomatizable by universal sentences.
Let $\mathbb{A} \models T$, $\mathbb{B} \leq \mathbb{A}$, and let $(\forall \mathbf{x})\varphi(\mathbf{x})$ be a universal axiom for T.
$\mathbb{A} \models (\forall \mathbf{x})\varphi(x)$ iff $\mathbb{A} \models \varphi[v]$ for every valuation in A, which implies $\mathbb{A} \models \varphi[v]$ for valuations with range in B, which implies $\mathbb{B} \models \varphi[v]$ for all valuations in B, which implies $\mathbb{B} \models (\forall \mathbf{x})\varphi(\mathbf{x})$. Hence $\mathbb{B} \models T$.

Proof. ($\Rightarrow$) In this direction we argue that if the class of models of T is closed under the formation of submodels, then T has a universal axiomatization.

Let Δ be the class of all L-sentences logically equivalent to universal sentences. Δ is closed under finite disjunction. According to the technical lemma, we must show that if $\mathbb{A}, \mathbb{B}$ are L-structures, $\mathbb{A} \models T$, and $(\forall \sigma \in \Delta)((\mathbb{A} \models \sigma) \quad \text{implies} \quad (\mathbb{B} \models \sigma))$, then $\mathbb{B} \models T$.

Claim. $T \cup \mathrm{Diag}_{\mathrm{at}}(\mathbb{B})$ is consistent.

Otherwise there exist $\pm$ atomic $\varphi_1(\mathbf{b}), \ldots, \varphi_n(\mathbf{b})$ such that $T \models \neg \bigwedge \varphi_i(\mathbf{b})$. Hence $T \models \neg(\exists \mathbf{x}) \bigwedge \varphi_i(\mathbf{x})$. Hence $T \models (\forall \mathbf{x}) \neg \bigwedge \varphi_i(\mathbf{x}) \in \Delta$. Hence if $\mathbb{A} \models (\forall \mathbf{x}) \neg \bigwedge \varphi_i(\mathbf{x})$, then $\mathbb{B} \models (\forall \mathbf{x}) \neg \bigwedge \varphi_i(\mathbf{x})$, or $\mathbb{B} \models \neg(\exists \mathbf{x}) \bigwedge \varphi_i(\mathbf{x})$. Which is false. $\square$ (Claim)

Let $\mathbb{C}_B$ be a model of $T \cup \mathrm{Diag}_{\mathrm{at}}(\mathbb{B})$. $\mathbb{C} \models T$, and $\mathbb{B} \leq \mathbb{C}$, so $\mathbb{B} \models T$. The technical lemma finishes the proof. $\square$

Corollaries

Corollary. (Existential in place of universal)
The class of models of T is closed under the formation of superstructures (or 'extensions') if and only if T has a set of axioms that are existential sentences. (I.e., Σ_1-sentences.)

Corollary. (Sentences in place of theories)
A single sentence σ is logically equivalent to a single universal sentence iff the validity of σ is preserved when passing to substructures.

Proof. Let $T = \{\sigma\}$ (or $\{\sigma\}^{\perp\perp}$). By the Theorem, the fact that the class of models of σ is closed under the formation of substructures implies that T has a set of universal axioms, which (by Compactness) may be taken to be finite: $\{(\forall \mathbf{x})\alpha_1(\mathbf{x}), \ldots, (\forall \mathbf{x})\alpha_n(\mathbf{x})\}^{\perp\perp} = \{\sigma\}^{\perp\perp}$. We may standardize the variables apart: $\{(\forall \mathbf{x}_1)\alpha_1(\mathbf{x}_1), \ldots, (\forall \mathbf{x}_n)\alpha_n(\mathbf{x}_n)\}$. Then σ has the same models as

$$(\forall \mathbf{x}_1) \cdots (\forall \mathbf{x}_n)\left(\bigwedge \alpha_1(\mathbf{x}_i)\right). \square$$