Practice with structures.

Recall that we defined a plane geometry to be a 2-sorted structure of the form

$$\Pi = \langle \underbrace{\mathcal{P},\mathcal{L}}_{\text{``sorts''}};\quad \underbrace{\mathcal{I},\mathcal{B},\overline{\mathcal{C}},\mathcal{C}_{\lhd}}_{\text{``structural elements''}} \rangle,$$

where
- (i) $\mathcal{P}$ is a set (of "points"),
- (ii) $\mathcal{L}$ is a set (of "lines"),
- (iii) $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$ is a binary relation (incidence),
- (iv) $\mathcal{B} \subseteq \mathcal{P} \times \mathcal{P} \times \mathcal{P} = \mathcal{P}^3$ is a ternary relation (betweenness),
- (v) $\overline{\mathcal{C}} \subseteq \mathcal{P}^4$ is a 4-ary relation, and
- (vi) $\mathcal{C}_{\lhd} \subseteq \mathcal{P}^6$ is a 6-ary relation,

and (as part of the definition of "plane") some axioms are satisfied.

We can write the first axiom of Euclidean geometry as a formal sentence. This axiom asserts that any two distinct points determine a unique line. Formally,

$$\forall P \forall Q ((P \neq Q) \to \exists \ell ((\mathcal{I}(P,\ell) \wedge \mathcal{I}(Q,\ell)) \wedge \forall m ((\mathcal{I}(P,m) \wedge \mathcal{I}(Q,m)) \to (m = \ell))))$$

Although we are not very far into the subject yet, this exercise asks that you show that certain common mathematical objects can be defined so that they are "structures" satisfying some axioms.

(1) Represent each of the following types of objects as "structures" of some kind. (This means: identify the "sorts" of elements the structure has and the types of "structural elements" needed to discuss the structure.)

    (a) Objects = planes. As structures: $\langle \underbrace{\mathcal{P},\mathcal{L}}_{\text{``sorts''}};\quad \underbrace{\mathcal{I},\mathcal{B},\overline{\mathcal{C}},\mathcal{C}_{\lhd}}_{\text{``structural elements''}} \rangle$,

    (b) Objects = groups. As structure:

    (c) Objects = vector spaces over the real scalar field. As structures:

    (d) Objects = graphs. As structures:

    (e) Objects = partially ordered sets. As structures:

    (f) Object = the natural numbers. As a structure:

(2) Write some formal sentences down which make sense for your structures, e.g., some of the axioms that define the objects they were invented to model.

    Suggestions:

    (a) Write a sentence that makes sense for any group which expresses that the group has a non-identity central element.

(b) Write a sentence that makes sense for any group which expresses that the group has exactly three elements.

(c) Write a sentence which makes sense for any poset that expresses that, between any two distinct comparable elements, there exists a third element.

(3) The real numbers, as an ordered field, is expressible as a structure
$$\langle \mathbb{R}; +, -, 0, \cdot, 1, < \rangle$$

(a) Write a formula $\varphi(x, y)$, in which x and y are not bound by any quantifiers, which expresses the relation "$y = |x|$".

(b) Write a sentence expressing that the function $f(x) = x^2$ is continuous at $x = 1$.

(4) Suppose that $\mathbb{P}$ is a partially ordered set, and you are told that it satisfies a sentence of the form

$(\forall x_1)(\exists x_2)(\forall x_3)(\forall x_4) \cdots (Q x_n)(x_1 < x_2 < x_3 < x_4 < \cdots < x_n), \quad Q \in \{\forall, \exists\}.$

How large can n be?