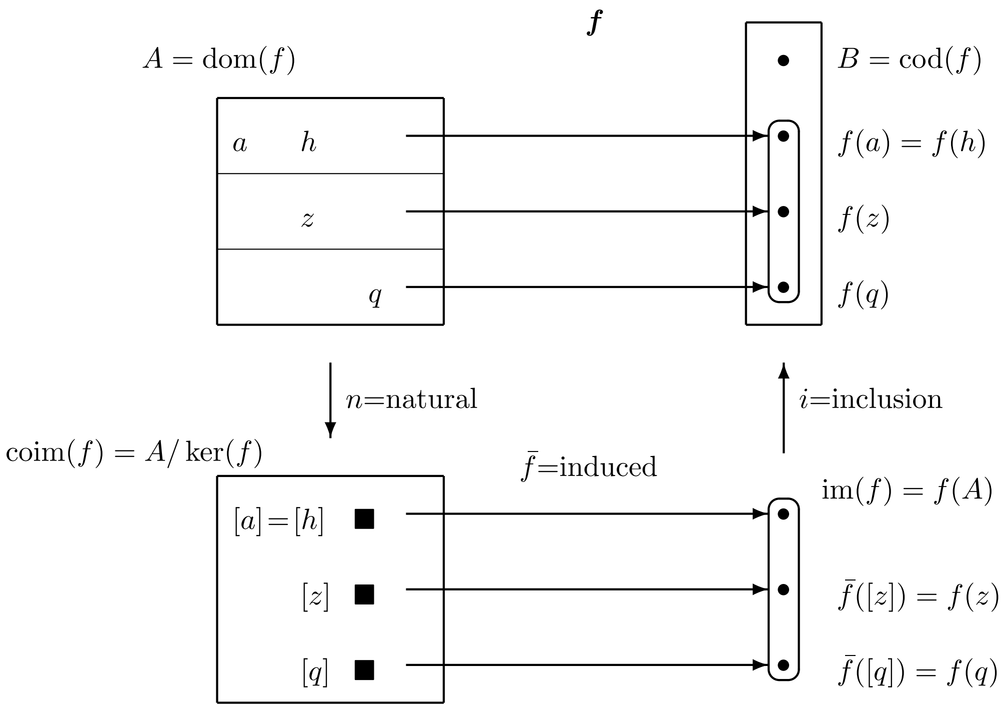
Morphisms and related concepts.



(1) The preimage of a singleton $\{b\}$ is written $f^{-1}(b)$ and sometimes called the *fiber* of f *over* b. The fiber containing the element a is sometimes written $[a]$.
(2) The *coimage* of f is the set $\mathrm{coim}(f) = \{f^{-1}(b) \ : \ b \in \mathrm{im}(f)\}$ of all nonempty fibers.
(3) The *kernel* of f is $\ker(f) = \{(a, a') \in A^2 \ : \ f(a) = f(a')\}$.
(4) The *natural map* is $n \colon A \to \mathrm{coim}(f) \colon a \mapsto [a]$. It is a surjection.
(5) The *inclusion map* is $i \colon \mathrm{im}(f) \to B \colon b \mapsto b$. It is an injection.
(6) The *induced map* is $\bar{f} \colon \mathrm{coim}(f) \to \mathrm{im}(f) \colon [a] \mapsto f(a)$. It is a bijection.

Definition 1. Let $\mathbb{A}$ and $\mathbb{B}$ be L-structures.

(1) A function $h : A \to B$ is a *homomorphism* if
- $h(c^{\mathbb{A}}) = c^{\mathbb{B}}$ for every constant symbol c,
- $F^{\mathbb{A}}(a_1, \ldots, a_n) = a$ implies $F^{\mathbb{B}}(h(a_1), \ldots, h(a_n)) = h(a)$ for every function symbol F, and
- $R^{\mathbb{A}}(a_1, \ldots, a_n) = \top$ implies $R^{\mathbb{B}}(h(a_1), \ldots, h(a_n)) = \top$ for every predicate symbol R.

(2) A homomorphism $h : \mathbb{A} \to \mathbb{B}$ is an *isomorphism* if there is a $g : \mathbb{B} \to \mathbb{A}$ such that $g \circ f = id_{\mathbb{A}}$ and $f \circ g = id_{\mathbb{B}}$.
(3) An isomorphism from $\mathbb{A}$ to itself is an *automorphism*.
(4) The *image* of a homomorphism $h : \mathbb{A} \to \mathbb{B}$, $\mathrm{im}(h)$, is the subset $h(A) \subseteq B$ and also is the substructure $\langle I; \{F^{\mathbb{B}}|_S\}, \{R^{\mathbb{B}}|_S\}, \{c^{\mathbb{B}}\}\rangle$ of $\mathbb{B}$, where $I = h(A)$.

(5) The *kernel* of h is the equivalence relation $\{(a,a') \in A^2 \mid h(a) = h(a')\}$.
(6) An *embedding* is a homomorphism that is an isomorphism with its image.
(7) An *elementary embedding* is a homomorphism $h : \mathbb{A} \to \mathbb{B}$ such that for any formula $\varphi(x_1,\ldots,x_n)$ it is the case that $\mathbb{A} \models \varphi(a_1,\ldots,a_n)$ implies $\mathbb{B} \models \varphi(b_1,\ldots,b_n)$. (Alternative notation: for every valuation v in $\mathbb{A}$ we have that $\mathbb{A} \models \varphi[v]$ implies $\mathbb{B} \models \varphi[h \circ v]$.)

Proposition 2. *An equivalence relation θ on $\mathbb{A}$ is the kernel of a homomorphism if and only if it is a* congruence*, which means that is is compatible with every function symbol in the sense that*

$$\begin{array}{rcll} a_1 & \equiv & a_1' & \pmod{\theta} \\ a_2 & \equiv & a_2' & \pmod{\theta} \\ & \vdots & & \\ a_n & \equiv & a_n' & \pmod{\theta} \\ \hline \Rightarrow\ F^{\mathbb{A}}(a_1,\ldots,a_n) & \equiv & F^{\mathbb{A}}(a_1',\ldots,a_n') & \pmod{\theta} \end{array}$$

If θ is a congruence on $\mathbb{A}$, then it is the kernel of the natural map

$$n : \mathbb{A} \to \mathbb{A}/\theta : a \mapsto a/\theta$$

of $\mathbb{A}$ onto the quotient $\mathbb{A}/\theta$*. The latter structure is defined to have*

- *universe A/θ,*
- $c^{\mathbb{A}/\theta} = c^{\mathbb{A}}/\theta$,
- $F^{\mathbb{A}/\theta}(a_1/\theta,\ldots,a_n/\theta) = F^{\mathbb{A}}(a_1,\ldots,a_n)/\theta$*, and*
- $R^{\mathbb{A}/\theta}(a_1/\theta,\ldots,a_n/\theta) = \top$ *provided* $\exists(a_1',\ldots,a_n')$ *such that* $(a_1/\theta,\ldots,a_n/\theta) = (a_1'/\theta,\ldots,a_n'/\theta)$ *and* $R^{\mathbb{A}}(a_1',\ldots,a_n') = \top$.

The category of L-structures is the category whose objects are the L-structures and whose morphisms are the homomorphisms. This category has products.

In general, a product object $\mathbb{P}$ for a family $\{\mathbb{A}_i \mid i \in I\}$ is an object equipped with *projection morphisms* $(\pi_i)_{i\in I}$, which are morphisms $\pi_i : \mathbb{P} \to \mathbb{A}_i$, and with the property that homomorphisms $h : \mathbb{B} \to \mathbb{P}$ are in 1-1 correspondence with sequences of maps into the coordinate structures:

$$h \in \mathrm{Hom}(\mathbb{B},\mathbb{P}) \quad \text{iff} \quad (\pi_i \circ h)_{i\in I} \in \prod_{i\in I} \mathrm{Hom}(\mathbb{B}, A_i).$$

The (Cartesian) product of L-structures $\mathbb{A}_i$ may be defined to have

- universe $P = \prod_{i\in I} A_i$,
- $c^{\mathbb{P}} = (c^{\mathbb{A}_i})_{i\in I}$,
- $F^{\mathbb{P}}(\mathbf{a}_1,\ldots,\mathbf{a}_n) = (F^{\mathbb{A}_i}((\mathbf{a}_1)_i,\ldots,(\mathbf{a}_n)_i)_{i\in I}$, ($F$ acts coordinatewise)
- $R^{\mathbb{P}}(\mathbf{a}_1,\ldots,\mathbf{a}_n) = \top$ iff $R^{\mathbb{A}_i}((\mathbf{a}_1)_i,\ldots,(\mathbf{a}_n)_i) = \top$ for all i. (R is true at a tuple iff it is true coordinatewise)

If $\mathbb{P}$ is the (Cartesian) product of L-structures $\{\mathbb{A}_i \mid i \in I\}$, then the Cartesian projections $\pi_j : \mathbb{A}_i \to A_j : (a_i) \mapsto a_j$ are morphisms, and with this family of morphisms the Cartesian product becomes a product object in the category of L-structures.