Łos's Theorem.

Our goal is to prove Łos's Theorem, which asserts that a formula is satisfied by a tuple in an ultraproduct iff it is satisfied in almost every coordinate modulo $\mathcal{U}$. In order to compare satisfaction in the ultraproduct with satisfaction in a coordinate structure we refer the following diagram:

$$\begin{array}{ccccc} X = \{x_1, x_2, \ldots\} & \xrightarrow{\ v\ } & \prod_{i \in I} \mathbb{A}_i & \xrightarrow{\ n\ } & \prod_{\mathcal{U}} \mathbb{A}_i = \mathbb{B} \\ & & \Big\downarrow{\scriptstyle \pi_j} & & \\ & & \mathbb{A}_j & & \end{array}$$

Here v is a valuation in the product $\prod_{i \in I} \mathbb{A}_i$, n is the natural quotient map onto the ultraproduct $\mathbb{B}$, and π_j is the j-th coordinate projection. Since n and π_j are surjective, any valuation in $\mathbb{B}$ or $\mathbb{A}_j$ factors through n or π_j respectively. Thus we can compare valuations in $\mathbb{B}$ and $\mathbb{A}_j$ via valuations in $\prod_{i \in I} \mathbb{A}_i$.

Theorem 1. *(Łos's Theorem) Let $\{\mathbb{A}_i \mid i \in I\}$ be a set of $\mathcal{L}$-structures and let $\mathcal{U}$ be an ultrafilter on I. Let $\mathbb{B} = \prod_{\mathcal{U}} \mathbb{A}_i$ be the ultraproduct. If $v \colon X \to \prod_{i \in I} A_i$ is a valuation, then for every formula $\varphi(\bar{x})$ it is the case that*

$$\mathbb{B} \models \varphi[n \circ v] \quad \textit{iff} \quad \{i \in I \mid \mathbb{A}_i \models \varphi[\pi_i \circ v]\} \in \mathcal{U}.$$

Proof. The displayed line in the theorem statement is proved by induction on the complexity of φ, which we may assume is built up from atomic formulas using $\neg, \wedge, \exists$.

Claim 2. *(Terms) For any term t, $t^{\mathbb{B}}[n \circ v] = [\langle t^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I \rangle]_{\theta_{\mathcal{U}}}$.*

- $(t = x_k)$

$$t^{\mathbb{B}}[n \circ v] = x_k[n \circ v] = n \circ v(x_k) = [v(x_k)]_{\theta_{\mathcal{U}}} = [\langle x_k[\pi_i \circ v] \mid i \in I \rangle]_{\theta_{\mathcal{U}}} = [\langle t^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I \rangle]_{\theta_{\mathcal{U}}}$$

- $(t = c)$

$$t^{\mathbb{B}}[n \circ v] = c^{\mathbb{B}}[n \circ v] = c^{\mathbb{B}} = [\langle c^{\mathbb{A}_i} \mid i \in I \rangle]_{\theta_{\mathcal{U}}} = [\langle c^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I \rangle]_{\theta_{\mathcal{U}}} = [\langle t^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I \rangle]_{\theta_{\mathcal{U}}}$$

- $(t = F(t_1, \ldots, t_m))$

$$\begin{aligned} t^{\mathbb{B}}[n \circ v] &= F^{\mathbb{B}}(t_1^{\mathbb{B}}[n \circ v], \ldots, t_m^{\mathbb{B}}[n \circ v]) = F^{\mathbb{B}}([\langle t_1^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I \rangle]_{\theta_{\mathcal{U}}}, \ldots, [\langle t_m^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I \rangle]_{\theta_{\mathcal{U}}}) \\ &= [\langle F^{\mathbb{A}_i}(t_1^{\mathbb{A}_i}[\pi_i \circ v], \ldots, t_m^{\mathbb{A}_i}[\pi_i \circ v]) \mid i \in I \rangle]_{\theta_{\mathcal{U}}} = [\langle t^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I \rangle]_{\theta_{\mathcal{U}}} \end{aligned}$$

Claim 3. *(Atomic formulas)*

- $(s = t)$

$$\begin{array}{rcl} \mathbb{B} \models (s = t)[n \circ v] & \leftrightarrow & s^{\mathbb{B}}[n \circ v] = t^{\mathbb{B}}[n \circ v] \\ & \leftrightarrow & [\langle s^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I \rangle]_{\theta_{\mathcal{U}}} = [\langle t^{\mathbb{A}_i}[\pi_i \circ v] \mid i \in I \rangle]_{\theta_{\mathcal{U}}} \\ & \leftrightarrow & \{ i \in I \mid s^{\mathbb{A}_i}[\pi_i \circ v] = t^{\mathbb{A}_i}[\pi_i \circ v] \} \in \mathcal{U} \\ & \leftrightarrow & \{ i \in I \mid \mathbb{A}_i \models (s = t)[\pi_i \circ v] \} \in \mathcal{U} \end{array}$$

- $(R(t_1, \ldots, t_m))$

$$\begin{array}{rcl} \mathbb{B} \models R(t_1, \ldots, t_m)[n \circ v] & \leftrightarrow & (t_1^{\mathbb{B}}[n \circ v], \ldots, t_m^{\mathbb{B}}[n \circ v]) \in R^{\mathbb{B}} \\ & \overset{\text{def}}{\leftrightarrow} & \{ i \in I \mid (t_1^{\mathbb{A}_i}[\pi_i \circ v], \ldots, t_m^{\mathbb{A}_i}[\pi_i \circ v]) \in R^{\mathbb{A}_i} \} \in \mathcal{U} \\ & \leftrightarrow & \{ i \in I \mid \mathbf{A}_i \models R(t_1, \ldots, t_m)[\pi_i \circ v] \} \in \mathcal{U} \end{array}$$

Claim 4. *(Connectives)*

- $(\neg)$

$$\begin{array}{rcl} \mathbb{B} \models \neg\varphi[n \circ v] & \leftrightarrow & \mathbb{B} \not\models \varphi[n \circ v] \\ & \leftrightarrow & \{ i \in I \mid \mathbb{A}_i \models \varphi[\pi_i \circ v] \} \notin \mathcal{U} \\ & \leftrightarrow & I \setminus \{ i \in I \mid \mathbb{A}_i \models \varphi[\pi_i \circ v] \} \in \mathcal{U} \\ & \leftrightarrow & \{ i \in I \mid \mathbb{A}_i \models \neg\varphi[\pi_i \circ v] \} \in \mathcal{U} \end{array}$$

- $(\wedge)$

$$\begin{array}{rcl} \mathbb{B} \models (\chi \wedge \varphi)[n \circ v] & \leftrightarrow & \mathbb{B} \models \chi[n \circ v] \text{ and } \mathbb{B} \models \varphi[n \circ v] \\ & \leftrightarrow & \{ i \in I \mid \mathbb{A}_i \models \chi[\pi_i \circ v] \} \in \mathcal{U} \text{ and } \{ i \in I \mid \mathbb{A}_i \models \varphi[\pi_i \circ v] \} \in \mathcal{U} \\ & \leftrightarrow & \{ i \in I \mid \mathbb{A}_i \models \chi[\pi_i \circ v] \} \cap \{ i \in I \mid \mathbb{A}_i \models \varphi[\pi_i \circ v] \} \in \mathcal{U} \\ & \leftrightarrow & \{ i \in I \mid \mathbb{A}_i \models (\chi \wedge \varphi)[\pi_i \circ v] \} \in \mathcal{U} \end{array}$$

Claim 5. *$(\exists)$*

$[\Rightarrow]$

$$\begin{array}{rcl} \mathbb{B} \models \exists x_k \varphi[n \circ v] & \longrightarrow & \text{there is a valuation } v' \equiv_k v \text{ such that } \mathbb{B} \models \varphi[n \circ v'] \\ & \longrightarrow & \{ i \in I \mid \mathbb{A}_i \models \varphi[\pi_i \circ v'] \} \in \mathcal{U} \\ & \longrightarrow & \{ i \in I \mid \mathbb{A}_i \models \exists x_k \varphi[\pi_i \circ v] \} \in \mathcal{U} \ \ (\text{since } \pi_i \circ v \equiv_k \pi_i \circ v') \end{array}$$

$[\Leftarrow]$ Assume that $\{ i \in I \mid \mathbb{A}_i \models \exists x_k \varphi[\pi_i \circ v] \} = U \in \mathcal{U}$. For each $i \in U$ pick a valuation $w_i \equiv_k \pi_i \circ v$ such that $\mathbb{A}_i \models \varphi[w_i]$. Choose any valuation $v' \colon X \to \prod \mathbb{A}_i$ such that $v' \equiv_k v$ and $\pi_i \circ v' = w_i$ when $i \in U$. Then $\{ i \in I \mid \mathbb{A}_i \models \varphi[\pi_i \circ v'] \}$ contains U, so $\mathbb{B} \models \varphi[n \circ v']$ by induction, so $\mathbb{B} \models \exists x_k \varphi[n \circ v]$. □