The lattice of L-theories, and its spectrum.

The Galois connection induced by $\models$ yields a closure operator on the class of L-sentences and a closure operator on the class of L-structures. A Galois-closed set of sentences is called a (closed) *theory*, while a Galois-closed set of structures is called an *elementary class*. The collection of L-theories forms a lattice under inclusion, and the collection of elementary classes also forms a lattice under inclusion; these lattices are dual to each other, and the Galois connection determines a dual isomorphism between them ($T \mapsto T^{\perp}$ and $\mathcal{K} \mapsto \mathcal{K}^{\perp}$).

The lattice of theories, $\mathcal{L}_{\mathrm{Th}}$ has these properties:

(1) It is algebraic. (It is *complete* and *compactly generated*. Completeness means that the meet or join of any subset of $\mathcal{L}_{\mathrm{Th}}$ exists. Compact generation means that every element is the join of the compact elements below it.)
(2) It is distributive. ($x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.)
(3) It is coatomistic. (Every element is the meet of coatoms.)
(4) Its compact elements are complemented. (If $x \in \mathcal{L}_{\mathrm{Th}}$ is compact, then there is a compact element y such that $x \wedge y = 0$ and $x \vee y = 1$.)

The important facts to note are that

(I) Any algebraic, distributive lattice is a *frame*. (I.e., satisfies $x \wedge (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} x \wedge y_i$ for any I.)
(II) A coatomistic frame is *spatial*. (That is, it is isomorphic to the lattice of open sets of a topological space.)
(III) An algebraic, spatial frame is isomorphic to the lattice of open sets of a topological space which has a basis of compact open sets. If, in addition, the compact elements of the frame are complemented, then the space has a basis of clopen sets.

Thus, the lattice $\mathcal{L}_{\mathrm{Th}}$ encodes a topological space, which is called the *spectrum* of the frame.

- Points of the spectrum = the meet irreducible elements of $\mathcal{L}_{\mathrm{Th}}$ = the complete L-theories.
- Closed sets of the spectrum have the form

$$V(\Sigma) = \{T \text{ complete} \mid \Sigma \subseteq T\}$$

  where Σ is a set of L-sentences.
- Basic clopen sets of the spectrum = sets of the form

$$O_\sigma = \{T \text{ complete} \mid \sigma \in T\} = V(\sigma) = V(\neg\sigma)^c$$

  where σ is an L-sentence

We will call the spectrum of $\mathcal{L}_{\mathrm{Th}}$ the *"space of complete L-theories"*.

Theorem 1. *The space of complete L-theories is*

- *compact,*
- *Hausdorff, and*
- *totally disconnected.*

I.e., it is a Stone space.

Glossary of new terms.

(1) **lattice.** The word "lattice" is applied to an algebraic structure $\langle L; \vee, \wedge\rangle$ with 2 binary operations, satisfying axioms that express that there is a partial order $\leq$ on L such that $x \vee y$ is the least upper bound (join) of x and y with respect to $\leq$ and $x \wedge y$ is the greatest lower bound (meet) of x and y with respect to $\leq$. The word "lattice" is also applied to the corresponding relational structure $\langle L; \leq\rangle$. The definitions guarantee that $x \leq y$ iff $x \wedge y = x$ iff $x \vee y = y$.

(2) **bounded lattice.** A lattice is bounded if it has a least element 0 and a greatest element 1.

(3) **complete lattice.** A lattice $\langle L; \leq\rangle$ is complete if any subset $U \subseteq L$ has a meet and a join. This includes the empty set. The empty join is the least element of L and the empty meet is the largest element of L.

(4) **compact element.** An element $c \in L$ compact if $c \leq \bigvee_{i \in I} d_i$ implies that there is a finite subset $I_0 \subseteq I$ such that $c \leq \bigvee_{i \in I_0} d_i$. (Any cover of c has a finite subcover.)

(5) **compactly generated lattice.** A lattice is compactly generated if every element is the join of the compact elements below it.

(6) **algebraic lattice.** A lattice is algebraic if it is complete and compactly generated.

(7) **distributive law.** The distributive law if the universally quantified equation $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

(8) **covering pair.** x is covered by y (written $x \prec y$) if $x < y$ and there is no z such that $x < z < y$.

(9) **atom/coatom.** An atom of a bounded lattice is an element a such that $0 \prec a$. A coatom is an element c such that $c \prec 1$.

(10) **atomistic/coatomistic lattice.** A lattice is atomistic if every element is a join of atoms. It is coatomistic if every element is a meet of coatoms.

(11) **complement of an element in a lattice.** u is a complement of v in L if $u \wedge v = 0$ and $u \vee v = 1$.

(12) **frame.** A frame is a complete lattice satisfying the infinite distributive law $x \wedge (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} x \wedge y_i$.

(13) **prime/irreducible/point in a frame.** An element $p \in L$ is meet-irreducible if $(x \wedge y = p) \to (x = p) \vee (y = p)$. Such elements are also called "prime", "irreducible", or "a point".

(14) **spatial frame.** A frame is spatial if it "has enough points", which means any two distinct elements can be separated by a point. Specifically, if $x, y \in L$ and $x \not\leq y$, then there exists a point p such that $y \leq p$ and $x \not\leq p$.