

$\mathcal{L}_{\kappa,\lambda}$

Let κ and λ be infinite cardinals. (There is no harm in assuming $\kappa \geq \lambda$.)

Definition 1. (The formulas of $\mathcal{L}_{\kappa,\lambda}$) Let σ be a finitary signature.

- (1) (Alphabet of symbols)
 - (a) variables: x_0, x_1, \dots (indexed by κ)
 - (b) nonlogical symbols of σ
 - (c) logical symbols: $=, \wedge, \neg, \exists$, as in first-order logic, together with
 - (i) $\bigwedge = (< \kappa)$ -fold conjunction, and
 - (ii) $\bigexists = (< \lambda)$ -fold quantification.

No harm in including \vee, \forall, \bigvee and \bigforall , although they are definable from the above.

- (2) (Preformulas) The set of $\mathcal{L}_{\kappa,\lambda}$ -preformulas is the smallest set \mathcal{P} that
 - (a) contains the atomic formulas of the first-order language determined by σ , and
 - (b) is closed under $\wedge, \vee, \neg, \exists, \forall, \bigwedge, \bigvee, \bigexists$, and \bigforall .
- (3) (Formulas) A $\mathcal{L}_{\kappa,\lambda}$ -formula is a $\mathcal{L}_{\kappa,\lambda}$ -preformula with $(< \lambda)$ -many free variables.

A basic example: If T is a first-order theory in a countable language and $p \in S_n(T)$, then $\Phi(\bar{x}) = \bigwedge_{\varphi \in p} \varphi(\bar{x})$ is a formula of $\mathcal{L}_{\omega_1, \omega}$.

Definition 2. (A fragment of $\mathcal{L}_{\omega_1, \omega}$) A subset $F \subseteq \mathcal{L}_{\omega_1, \omega}$ is a *fragment* if it

- (1) contains all first-order formulas,
- (2) is closed under $\wedge, \vee, \neg, \exists, \forall$,
- (3) is closed under the formation of subformulas, and
- (4) is closed under the operation of substituting terms for free variables.

If $F \subseteq \mathcal{L}_{\omega_1, \omega}$ is a fragment, then an F -type is a subset $p \subseteq F_{\bar{x}}$ for which there is a countable structure \mathbf{A} and a tuple \bar{a} in A such that

$$p = \{\varphi(\bar{x}) \in F \mid \mathbf{A} \models \varphi[\bar{a}]\}.$$

If $F \subseteq \mathcal{L}_{\omega_1, \omega}$, write $\mathbf{A} \equiv_F \mathbf{B}$ for F -elementary equivalence. and $S_n(F, T)$ for the space of F -types.

Exercises.

- (1) A sub-preformula of an $\mathcal{L}_{\omega_1, \omega}$ -formula is a formula. (I.e., it is not just a preformula.)
- (2) If σ is a countable signature, then $|\mathcal{L}_{\omega_1, \omega}| = 2^\omega$.
- (3) If $G \subseteq \mathcal{L}_{\omega_1, \omega}$ is a countable subset, then there is a countable fragment $F \subseteq \mathcal{L}_{\omega_1, \omega}$ such that $G \subseteq F$.