

Formulas.

In this note we say what is meant by a formal mathematical statement. We first begin by specifying a *language* (called L), by which we mean specifying which predicate symbols ($\mathcal{P} = \{=, <, \dots\}$), which operation symbols ($\mathcal{O} = \{+, \cdot, -, \dots\}$), and which constant symbols ($\mathcal{C} = \{0, 1, \pi, \dots\}$) we need for the ideas we want to express.

Example 1. (1) The language of *set theory* has one predicate symbol \in , no operation symbols, and no constant symbols.
 (2) One language for *number theory* (i.e., the theory of the natural numbers) has one operation symbol, S (for successor), one constant symbol, 0 (for zero), and no non-logical predicate symbols.
 (3) One language for the real numbers has operation symbols $\mathcal{O} = \{+, \cdot, -\}$, constant symbols $\mathcal{C} = \{0, 1\}$, and predicate symbols $\mathcal{P} = \{<\}$.

Fixing L , we can define terms, atomic formulas, then arbitrary formulas in this language.

Definition 2. The set of all L -terms is the smallest set \mathcal{T} such that

- (i) \mathcal{T} contains all variables and constant symbols, and
- (ii) if $f \in \mathcal{O}$ is an n -ary operation symbol and $t_1, \dots, t_n \in \mathcal{T}$, then $f(t_1, \dots, t_n) \in \mathcal{T}$.

Example 3. (1) In the language of set theory the only terms are variables.
 (2) In the language of number theory whose nonlogical symbols are 0 and S , the only terms are of the form $S^k(0)$ and $S^k(x_i)$, $k = 0, 1, 2, \dots$.
 (3) In the language of the real numbers whose nonlogical symbols are $+, \cdot, -, 0, 1, <$ there are very complicated terms like $((x_1 \cdot x_{17}) + ((x_1 \cdot 0) \cdot x_9)) + 1$.

Definition 4. The set of all *atomic L -formulas* is the set of all strings $P(t_1, \dots, t_n)$ where P is an n variable predicate symbol and the t_i are terms.

Example 5. (1) In the language of set theory the only atomic formulas are of the form $(x_i \in x_j)$.
 (2) In the language of number theory whose nonlogical symbols are 0 and S , the only atomic formulas are equations of the form $(S^k(x_i) = S^\ell(x_j))$, $(S^k(x_i) = S^\ell(0))$, $(S^k(0) = S^\ell(x_j))$, and $(S^k(0) = S^\ell(0))$.
 (3) In the language of the real numbers whose nonlogical symbols are $+, \cdot, -, 0, 1, <$ there are very complicated atomic formulas, including $(1 < (x \cdot x))$ or $((x_1 + (x_2 + x_3)) = ((x_1 + x_2) + x_3))$.

Definition 6. The set of all L -formulas is the smallest set \mathcal{F} such that

- (i) \mathcal{F} contains all atomic formulas, and
- (ii) if $\alpha, \beta \in \mathcal{F}$ and x is a variable, then the following are in \mathcal{F} : $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$, $(\neg \alpha)$, $(\forall x \alpha)$, $(\exists x \alpha)$.

Example 7. In any language, the formulas get complicated. Here are some examples.

- (1) (Set theory) We can express “ x is a subset of y ” with the formula $\alpha(x, y) = “\forall z ((z \in x) \rightarrow (z \in y))”$.
- (2) (Number theory) We can express that the successor function is 1-1 with the formula $\beta = \forall x \forall y ((S(x) = S(y)) \rightarrow (x = y))$.
- (3) (Real numbers) We can express that any monic cubic polynomial has a root with the formula $\gamma = \forall y_1 \forall y_2 \forall y_3 \exists x (x^3 + y_1 \cdot x^2 + y_2 \cdot x + y_3 = 0)$.

Exercises. Express the given fact or relation in the language whose nonlogical symbols are those given.

- (1) Express “There is a set with no elements” in the language of set theory.
- (2) Express “ x has exactly two elements” in the language of set theory.
- (3) Write the Axiom of Extensionality in the language of set theory.
- (4) One language for ordered sets has \leq as its only nonlogical symbol. In this language express “ x is not the largest element and not the smallest element.”
- (5) Express Fermat’s Last Theorem in a language for number theory whose nonlogical symbols are $0, +, \cdot, \wedge, <$. (Fermat’s Last Theorem is the statement that if x, y, z, n are nonzero natural numbers and n is at least 3, then $x^n + y^n = z^n$ does not hold.)