Formulas.

In this note we say what is meant by a formal mathematical statement. We first begin by specifying a *language* (called L), by which we mean specifying which predicate symbols ($\mathcal{P} = \{=, <, \ldots\}$), which operation symbols ($\mathcal{O} = \{+, \cdot, -, \ldots\}$), and which constant symbols ($\mathcal{C} = \{0, 1, \pi, \ldots\}$) we need for the ideas we want to express.

Example 1. (1) The language of *set theory* has one predicate symbol $\in$, no operation symbols, and no constant symbols.

(2) One language for *number theory* (i.e., the theory of the natural numbers) has one operation symbol, S (for successor), one constant symbol, 0 (for zero), and no nonlogical predicate symbols.

(3) One language for the real numbers has operation symbols $\mathcal{O} = \{+, \cdot, -\}$, constant symbols $\mathcal{C} = \{0, 1\}$, and predicate symbols $\mathcal{P} = \{<\}$.

Fixing L, we can define terms, atomic formulas, then arbitrary formulas in this language.

Definition 2. The set of all *L-terms* is the smallest set $\mathcal{T}$ such that

(i) $\mathcal{T}$ contains all variables and constant symbols, and

(ii) if $f \in \mathcal{O}$ is an n-ary operation symbol and $t_1, \ldots, t_n \in \mathcal{T}$, then $f(t_1, \ldots, t_n) \in \mathcal{T}$.

Example 3. (1) In the language of set theory the only terms are variables.

(2) In the language of number theory whose nonlogical symbols are 0 and S, the only terms are of the form $S^k(0)$ and $S^k(x_i)$, $k = 0, 1, 2, \ldots$.

(3) In the language of the real numbers whose nonlogical symbols are $+, \cdot, -, 0, 1, <$ there are very complicated terms like $(((x_1 \cdot x_{17}) + ((x_1 \cdot 0) \cdot x_9)) + 1)$.

Definition 4. The set of all *atomic L-formulas* is the set of all strings $P(t_1, \ldots, t_n)$ where P is an n variable predicate symbol and the t_i are terms.

Example 5. (1) In the language of set theory the only atomic formulas are of the form $(x_i \in x_j)$ or $(x_i = x_j)$.

(2) In the language of number theory whose nonlogical symbols are 0 and S, the only atomic formulas are equations of the form $(S^k(x_i) = S^\ell(x_j))$, $(S^k(x_i) = S^\ell(0))$, $(S^k(0) = S^\ell(x_j))$, and $(S^k(0) = S^\ell(0))$.

(3) In the language of the real numbers whose nonlogical symbols are $+, \cdot, -, 0, 1, <$ there are very complicated atomic formulas, including $(1 < (x \cdot x))$ or $((x_1 + (x_2 + x_3)) = ((x_1 + x_2) + x_3))$.

Definition 6. The set of all *L-formulas* is the smallest set $\mathcal{F}$ such that

(i) $\mathcal{F}$ contains all atomic formulas, and

(ii) if $\alpha, \beta \in \mathcal{F}$ and x is a variable, then the following are in $\mathcal{F}$: $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \to \beta)$, $(\alpha \leftrightarrow \beta)$, $(\neg \alpha)$, $(\forall x\ \alpha)$, $(\exists x\ \alpha)$.

Example 7. In any language, the formulas get complicated. Here are some examples.

(1) (Set theory) We can express "x is a subset of y" with the formula $\alpha(x, y) =$ "$\forall z\ ((z \in x) \rightarrow (z \in y))$".
(2) (Number theory) We can express that the successor function is 1-1 with the formula $\beta = \forall x\ \forall y\ ((S(x) = S(y)) \rightarrow (x = y))$.
(3) (Real numbers) We can express that any monic cubic polynomial has a root with the formula $\gamma = \forall y_1\ \forall y_2\ \forall y_3\ \exists x\ (x^3 + y_1 \cdot x^2 + y_2 \cdot x + y_3 = 0)$.

Exercises. Express the given fact or relation in the language whose nonlogical symbols are those given.

(1) Express "There is a set with no elements" in the language of set theory.

(2) Express "x has exactly two elements" in the language of set theory.

(3) Write the Axiom of Extentionality in the language of set theory.

(4) One language for ordered sets has $\leq$ as its only nonlogical symbol. In this language express "x is not the largest element and not the smallest element."

(5) Express Fermat's Last Theorem in a language for number theory whose nonlogical symbols are $0, +, \cdot, \frown, <$. (Fermat's Last Theorem is the statement that if x, y, z, n are nonzero natural numbers and n is at least 3, then $x \frown n + y \frown n = z \frown n$ does not hold.)