The theories of the fields $\mathbb{C}$ and $\mathbb{R}_<$

April 22, 2020

Q.E. Theorems

Recall: A theory T has q.e. iff it has q.e. “locally”.
Having q.e. locally means that every type is determined by its q.f. part.

Theorem. (cf. Marker 3.1.4)
T has q.e. iff for all $M, N \models T$,
if $\mathbf{a} \in M^n$, $\mathbf{b} \in N^n$, then
$\mathrm{tp}_M^{q.f.}(\mathbf{a}) = \mathrm{tp}_N^{q.f.}(\mathbf{b})$ implies $\mathrm{tp}_M(\mathbf{a}) = \mathrm{tp}_N(\mathbf{b})$. $\square$

But how do you check this property of types?

Algebraically prime extensions

A theory T has *algebraically prime extensions* if whenever A is a substructure of a model of T there exists an extension A^* of A, where $A^* \models T$, such that any embedding of A into a model of T extends to A^*.

T has algebraically prime extensions if whenever A is a substructure of a model of T, then $T \cup \mathrm{Diag}(A)$ has an *algebraically prime model*.

Examples and nonexamples

Examples.

1. The theory of fields in the language of rings ($\mathbb{F} = \langle F; +, -, 0, \cdot, 1\rangle$) has algebraically prime extensions. (A substructure of a field $\mathbb{F}$ of characteristic zero is an integral domain A of characteristic zero. E.g. $A = \mathbb{Z}$. The field of fractions A^* is algebraically prime over A. E.g. $A^* = \mathbb{Q}$.)
2. ACF_0 has algebraically prime extensions. (A^* is the algebraically closure of the field of fractions of A.)
3. DAG has algebraically prime extensions. (A^* is the divisible hull of A.)

Nonexample.

1. The theory $T = \mathrm{Th}(\mathbb{R})$ does not have algebraically prime extensions. Reason: $A = \mathbb{R}(t)$ is embeddable in a model of T, but any A^* would have to decide which of $t, -t$ should be a square.

Theorem. If T

1. has algebraically prime extensions, and
2. is model complete,

then T has q.e.

Proof. Choose models $M, N \models T$ and tuples $\mathbf{a} \in M^n$ and $\mathbf{b} \in N^n$.
Let A be the substructure of M generated by the elements of $\mathbf{a}$,
and let B be the substructure of N generated by the elements of $\mathbf{b}$.
Let A^* and B^* be algebraically prime extensions of A and B.
$\mathrm{tp}_M^{q.f.}(\mathbf{a}) = \mathrm{tp}_N^{q.f.}(\mathbf{b}) \Rightarrow$
$\exists f : A \xrightarrow{\sim} B : \mathbf{a} \mapsto \mathbf{b} \Rightarrow$
$\mathrm{tp}_{A^*}(\mathbf{a}) = \mathrm{tp}_{B^*}(\mathbf{b})$.
But $A^* \prec M$, $B^* \prec N$ since all are models of T and T is model complete,
hence $\mathrm{tp}_M(\mathbf{a}) = \mathrm{tp}_{A^*}(\mathbf{a}) = \mathrm{tp}_{B^*}(\mathbf{b}) = \mathrm{tp}_N(\mathbf{b})$. $\square$

A refinement

In the previous theorem, we can assume less than "T is model complete", although in the end T will have to be model complete by q.e.

Theorem. (cf. Marker 3.1.12)
If T

1. has algebraically prime extensions, and
2. whenever $M, N \models T$ and $M \leq N$, then M is *existentially closed* in N,

then T has q.e.

Here M is existentially closed in the extension N if whenever $\mathbf{a} \in M^n$ and $\varphi(\mathbf{x}, y)$ is primitive + q.f., then $N \models (\exists y)\varphi(\mathbf{a}, y)$ implies $M \models (\exists y)\varphi(\mathbf{a}, y)$.

ACF_0

"Easy" Fact from Algebra. ACF_0 has algebraically prime extensions. This follows from the facts that every integral domain A has a field of fractions, which has an algebraic closure A^*, and the extension property holds.

Less-easy Fact. If $M \leq N$ are models of ACF_0, then M is existentially closed in N.

If $\varphi(\mathbf{x}, y)$ is primitive+q.f., ($\bigwedge \pm$atomic)
then $\varphi(\mathbf{a}, y)$ is $\bigwedge(p(\mathbf{a}, y) = 0) \wedge \bigwedge(q(\mathbf{a}, y) \neq 0)$.
If p-part is not empty, then $N \models (\exists y)\varphi(\mathbf{a}, y)$ implies $M \models (\exists y)\varphi(\mathbf{a}, y)$ because M contains all roots of the p-part that lie in N.
If p-part is empty, then $M \models (\exists y)\varphi(\mathbf{a}, y)$ because M is infinite while zero sets of 1-variable polynomials are finite. $\square$

The same strategy (with more algebra) proves that $\mathbb{R}_<$ has q.e. That is, it is more work to prove that $\mathrm{Th}(\mathbb{R}_<)$ has algebraically prime extensions, and there are more primitive + q.f. formulas to consider.