The theories of the fields $\mathbb{C}$ and $\mathbb{R}_<$

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But how do you check this property of types?

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Equivalently, T has algebraically prime extensions if whenever A is a substructure of a model of T, then the theory $T \cup \mathrm{Diag}(A)$ has a model embeddable in all other models of this theory.

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1. The theory $T = \text{Th}(\mathbb{R})$ does not have algebraically prime extensions. Reason: $A = \mathbb{R}(t)$ is embeddable in a model of T, but any A^* would have to decide which of $t, -t$ should be a square.

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Theorem. If T

1. has algebraically prime extensions, and
2. is model complete,

then T has q.e.

Proof. Choose models $M, N \models T$ and tuples $\mathbf{a} \in M^n$ and $\mathbf{b} \in N^n$.
Let A be the substructure of M generated by the elements of $\mathbf{a}$,
and let B be the substructure of N generated by the elements of $\mathbf{b}$.
Let A^* and B^* be algebraically prime extensions of A and B.
$\mathrm{tp}_M^{q.f.}(\mathbf{a}) = \mathrm{tp}_N^{q.f.}(\mathbf{b}) \Rightarrow$
$\exists f : A \xrightarrow{\sim} B : \mathbf{a} \mapsto \mathbf{b} \Rightarrow$
$\exists f^* : A^* \hookrightarrow B^* : \mathbf{a} \mapsto \mathbf{b} \Rightarrow$
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Here M is existentially closed in the extension N if whenever $\mathbf{a} \in M^n$ and $\varphi(\mathbf{x}, y)$ is primitive + q.f., then $N \models (\exists y)\varphi(\mathbf{a}, y)$ implies $M \models (\exists y)\varphi(\mathbf{a}, y)$.

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Remarks.

1. If T has q.e., and $\mathbf{A}$ is a model of T, then the P-definable sets of $\mathbf{A}^n$ are the sets that are 'Boolean combinations' of sets P-definable by $\pm$atomic formulas.
2. In particular, if T is some theory of fields, T has q.e., and $\mathbb{F} \models T$ is infinite, then every P-definable subset of $\mathbb{F}$ is finite or cofinite. (A q.e.-theory of infinite fields is 'strongly minimal'.)
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