Quantifier elimination

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Things you already know

It is possible to express that

$$\varphi(a,b,c) : (\exists x)(ax^2 + bx + c = 0)$$

is true in the ordered field $\mathbb{R}$, for a given choice of a, b, c, in a different way that doesn't use a quantifier:

$$((a \neq 0) \wedge (b^2 - 4ac \geq 0)) \vee ((a = 0) \wedge ((b \neq 0) \vee (c = 0))).$$

It is possible to express that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$ has an inverse using quantifiers:

$$(\exists t)(\exists u)(\exists v)(\exists w) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} t & u \\ v & w \end{bmatrix} = \begin{bmatrix} t & u \\ v & w \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

or without:

$$ad - bc \neq 0.$$

Definition. A theory T has **quantifier elimination** if for every formula $\varphi(\mathbf{x})$ there is a quantifier-free formula $\alpha(\mathbf{x})$ such that

$$T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x})).$$

Examples.

1. The theory of infinite sets in the language of equality.
2. The theory of dense linear order in the language of ordered sets.
3. The theory of algebraically closed fields in the ring/field language.
4. The theory of the real numbers in the language of ordered fields.

Non-examples.

1. The theory of $\langle \mathbb{N}; \leq \rangle$ in the language of ordered sets.
2. The theory of the real numbers in the ring/field language.

Why do we care?

If T has q.e., then

1. all embeddings between models are elementary,
2. easier to check completeness: if there is a constant in the language, check whether the theory decides the truth of every quantifier-free sentence. If no constant, must also check whether the theory decides the truth of $(\exists x)\alpha(x)$ for every q.f., 1-variable, atomic formula $\alpha(x)$, where $T \models (\forall x)(\forall y)(\alpha(x) \leftrightarrow \alpha(y))$.
3. easier to establish ω-categoricity: any complete theory in a finite relational language, which has an infinite model, will be ω-categorical. (May not be true if T does not have q.e.!)
4. becomes easier to classify definable sets in models.

What's language got to do with it?

How to cheat.

The **Morleyization** (or **atomization**) of an L-theory T is the theory T' in an expanded language having a new relation symbol $R_\varphi(x_1, \ldots, x_n)$ for every L-formula $\varphi(x_1, \ldots, x_n)$ where we add to T the sentences

$$(\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow R_\varphi(\mathbf{x})).$$

It is obvious that the models of T and T' are essentially the same, and that T' has q.e.

This is of theoretical value, and typically doesn't bypass any practical complications.

The field of real numbers is an interesting example.

(Tarski-Seidenberg) Implies that the theory of the ordered field $\langle \mathbb{R}; +, -, 0, \cdot, 1, \leq \rangle$ has q.e.

(Macintyre) The only theories of fields with q.e. are the algebraically closed fields. In particular, $\langle \mathbb{R}; +, -, 0, \cdot, 1 \rangle$ does not have q.e.

Thus every use of a quantifier in a formula for $\mathbb{R}$ can be reduced to

$$\varphi_{\leq}(x, y): \qquad (\exists z)(y = x + z^2).$$

Methods for establishing q.e.

1. BRUTE FORCE!! (Requires analysis/classification of the q.f. formulas.)
2. $\approx \approx$ gentle $\approx$ persuasion $\approx \approx \approx \approx \approx \approx$

Brute force example

The theory T of infinite sets in the language of equality has q.e.

Any formula can be put in the form $(Q_1 x_{i_1}) \cdots (Q_n x_{i_n})(\bigvee \bigwedge \pm\text{atomic})$.

To eliminate Q's, it suffices to eliminate $\exists y$ from $(\exists y)(\bigvee \bigwedge \pm\text{atomic})$.

But $\exists$ distributes over $\bigvee$, so it suffices to eliminate $\exists y$ from $(\exists y)(\bigwedge \pm\text{atomic})$.
Such formulas are "primitive formulas".

"Arrangement" of variables: For some equivalence relation E on $\{1, \ldots, n\}$

$$\mathrm{Arr}_E(x_1, \ldots, x_n) = \bigwedge_{(i,j)\in E} (x_i = x_j) \wedge \bigwedge_{(i,j)\notin E} \neg(x_i = x_j).$$

This is a conjunction of $\pm$atomic.

Brute force example: more

Lemma. (Classification of q.f. formulas modulo T.) Any q.f. formula is either inconsistent (e.g. $\neg(x_i = x_i) \equiv \bot$) or is equivalent to a disjunction of finitely many arrangements. (I.e., any "partial arrangement" is a disjunction of "total arrangements".)

Hence, if $\varphi(\mathbf{x}, y)$ is q.f., $T \models (\forall \mathbf{x})((\exists y)\varphi(\mathbf{x}, y) \leftrightarrow (\exists y)(\bigvee_k \mathrm{Arr}_{E_k}(\mathbf{x}, y)))$.

Hence suffices to eliminate $\exists y$ in $(\exists y)\mathrm{Arr}_E(x_1, \ldots, x_n, y)$.

Let E^* be the restriction of E from $\{x_1, \ldots, x_n, y\}$ to $\{x_1, \ldots, x_n\}$. Then note that

$$T \models (\forall \mathbf{x})((\exists y)\mathrm{Arr}_E(\mathbf{x}, y) \leftrightarrow \mathrm{Arr}_{E^*}(\mathbf{x})).$$