Quantifier elimination, Part 2

Recall

Recall

Definition.

Recall

Definition. A theory T has **quantifier elimination** if for every formula $\varphi(\mathbf{x})$ there is a quantifier-free formula $\alpha(\mathbf{x})$ such that

Recall

Definition. A theory T has **quantifier elimination** if for every formula $\varphi(\mathbf{x})$ there is a quantifier-free formula $\alpha(\mathbf{x})$ such that

$$T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x})).$$

Recall

Definition. A theory T has **quantifier elimination** if for every formula $\varphi(\mathbf{x})$ there is a quantifier-free formula $\alpha(\mathbf{x})$ such that

$$T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x})).$$

If T has q.e., then

Recall

Definition. A theory T has **quantifier elimination** if for every formula $\varphi(\mathbf{x})$ there is a quantifier-free formula $\alpha(\mathbf{x})$ such that

$$T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x})).$$

If T has q.e., then

1. all embeddings between models are elementary,

Recall

Definition. A theory T has **quantifier elimination** if for every formula $\varphi(\mathbf{x})$ there is a quantifier-free formula $\alpha(\mathbf{x})$ such that

$$T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x})).$$

If T has q.e., then

1. all embeddings between models are elementary,

Recall

Definition. A theory T has **quantifier elimination** if for every formula $\varphi(\mathbf{x})$ there is a quantifier-free formula $\alpha(\mathbf{x})$ such that

$$T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x})).$$

If T has q.e., then

1. all embeddings between models are elementary,
2. it is easier to check completeness of T,

Recall

Definition. A theory T has **quantifier elimination** if for every formula $\varphi(\mathbf{x})$ there is a quantifier-free formula $\alpha(\mathbf{x})$ such that

$$T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x})).$$

If T has q.e., then

1. all embeddings between models are elementary,
2. it is easier to check completeness of T,

Recall

Definition. A theory T has **quantifier elimination** if for every formula $\varphi(\mathbf{x})$ there is a quantifier-free formula $\alpha(\mathbf{x})$ such that

$$T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x})).$$

If T has q.e., then

1. all embeddings between models are elementary,
2. it is easier to check completeness of T,
3. it is easier to establish $\aleph_0$-categoricity of some T's,

Recall

Definition. A theory T has **quantifier elimination** if for every formula $\varphi(\mathbf{x})$ there is a quantifier-free formula $\alpha(\mathbf{x})$ such that

$$T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x})).$$

If T has q.e., then

1. all embeddings between models are elementary,
2. it is easier to check completeness of T,
3. it is easier to establish $\aleph_0$-categoricity of some T's,

Recall

Definition. A theory T has **quantifier elimination** if for every formula $\varphi(\mathbf{x})$ there is a quantifier-free formula $\alpha(\mathbf{x})$ such that

$$T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x})).$$

If T has q.e., then

1. all embeddings between models are elementary,
2. it is easier to check completeness of T,
3. it is easier to establish $\aleph_0$-categoricity of some T's,
4. it is easier to classify definable sets in models.

Recall

Definition. A theory T has **quantifier elimination** if for every formula $\varphi(\mathbf{x})$ there is a quantifier-free formula $\alpha(\mathbf{x})$ such that

$$T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x})).$$

If T has q.e., then

1. all embeddings between models are elementary,
2. it is easier to check completeness of T,
3. it is easier to establish $\aleph_0$-categoricity of some T's,
4. it is easier to classify definable sets in models.

Recall

Definition. A theory T has **quantifier elimination** if for every formula $\varphi(\mathbf{x})$ there is a quantifier-free formula $\alpha(\mathbf{x})$ such that

$$T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x})).$$

If T has q.e., then

1. all embeddings between models are elementary,
2. it is easier to check completeness of T,
3. it is easier to establish $\aleph_0$-categoricity of some T's,
4. it is easier to classify definable sets in models.

We have shown that the theory of infinite sets has q.e. using "brute force".

Definition. A theory T has **quantifier elimination** if for every formula $\varphi(\mathbf{x})$ there is a quantifier-free formula $\alpha(\mathbf{x})$ such that

$$T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x})).$$

If T has q.e., then

1. all embeddings between models are elementary,
2. it is easier to check completeness of T,
3. it is easier to establish $\aleph_0$-categoricity of some T's,
4. it is easier to classify definable sets in models.

We have shown that the theory of infinite sets has q.e. using "brute force". Now we are going to develop other techniques to establish that a theory has q.e.

Atomic diagrams

Atomic diagrams

Definition.

Atomic diagrams

Definition. If $\mathbf{A}$ is an L-structure, then the **atomic diagram** of $\mathbf{A}$

Atomic diagrams

Definition. If $\mathbf{A}$ is an L-structure, then the **atomic diagram** of $\mathbf{A}$ is the L_A-theory axiomatized by the $\pm$atomic sentences that hold in $\mathbf{A}_A$.

Definition. If $\mathbf{A}$ is an L-structure, then the **atomic diagram** of $\mathbf{A}$ is the L_A-theory axiomatized by the $\pm$atomic sentences that hold in $\mathbf{A}_A$. (Write $\mathrm{Diag}(\mathbf{A})$ for this theory.)

Atomic diagrams

Definition. If $\mathbf{A}$ is an L-structure, then the **atomic diagram** of $\mathbf{A}$ is the L_A-theory axiomatized by the $\pm$atomic sentences that hold in $\mathbf{A}_A$. (Write $\mathrm{Diag}(\mathbf{A})$ for this theory.)

Diagram Lemma.

Atomic diagrams

Definition. If $\mathbf{A}$ is an L-structure, then the **atomic diagram** of $\mathbf{A}$ is the L_A-theory axiomatized by the $\pm$atomic sentences that hold in $\mathbf{A}_A$. (Write $\mathrm{Diag}(\mathbf{A})$ for this theory.)

Diagram Lemma. $\mathbf{B}_A$ is a model of the atomic diagram of $\mathbf{A}$ iff the mapping $h : \mathbf{A} \to \mathbf{B} : a^{\mathbf{A}} \to a^{\mathbf{B}}$ is an embedding.

Atomic diagrams

Definition. If $\mathbf{A}$ is an L-structure, then the **atomic diagram** of $\mathbf{A}$ is the L_A-theory axiomatized by the $\pm$atomic sentences that hold in $\mathbf{A}_A$. (Write $\mathrm{Diag}(\mathbf{A})$ for this theory.)

Diagram Lemma. B_A is a model of the atomic diagram of $\mathbf{A}$ iff the mapping $h : \mathbf{A} \to \mathbf{B} : a^{\mathbf{A}} \to a^{\mathbf{B}}$ is an embedding.

Proof.

Atomic diagrams

Definition. If $\mathbf{A}$ is an L-structure, then the **atomic diagram** of $\mathbf{A}$ is the L_A-theory axiomatized by the $\pm$atomic sentences that hold in $\mathbf{A}_A$. (Write $\text{Diag}(\mathbf{A})$ for this theory.)

Diagram Lemma. $\mathbf{B}_A$ is a model of the atomic diagram of $\mathbf{A}$ iff the mapping $h : \mathbf{A} \to \mathbf{B} : a^{\mathbf{A}} \to a^{\mathbf{B}}$ is an embedding.

Proof. If $\mathbf{B}_A$ is a model of the atomic diagram of $\mathbf{A}$, then $\mathbf{B}_A$ satisfies all L_A-sentences of the form $\pm(a_i = a_j)$, $\pm(F(a_{i_1}, \ldots, a_{i_n}) = a_j)$, $\pm R(a_{i_1}, \ldots, a_{i_n})$ that are true in $\mathbf{A}$.

Atomic diagrams

Definition. If $\mathbf{A}$ is an L-structure, then the **atomic diagram** of $\mathbf{A}$ is the L_A-theory axiomatized by the $\pm$atomic sentences that hold in $\mathbf{A}_A$. (Write $\mathrm{Diag}(\mathbf{A})$ for this theory.)

Diagram Lemma. $\mathbf{B}_A$ is a model of the atomic diagram of $\mathbf{A}$ iff the mapping $h : \mathbf{A} \to \mathbf{B} : a^{\mathbf{A}} \to a^{\mathbf{B}}$ is an embedding.

Proof. If $\mathbf{B}_A$ is a model of the atomic diagram of $\mathbf{A}$, then $\mathbf{B}_A$ satisfies all L_A-sentences of the form $\pm(a_i = a_j)$, $\pm(F(a_{i_1}, \ldots, a_{i_n}) = a_j)$, $\pm R(a_{i_1}, \ldots, a_{i_n})$ that are true in $\mathbf{A}$. From its definition, h must preserve the satisfaction of these sentences, hence h is an embedding.

Atomic diagrams

Definition. If $\mathbf{A}$ is an L-structure, then the **atomic diagram** of $\mathbf{A}$ is the L_A-theory axiomatized by the $\pm$atomic sentences that hold in $\mathbf{A}_A$. (Write $\mathrm{Diag}(\mathbf{A})$ for this theory.)

Diagram Lemma. $\mathbf{B}_A$ is a model of the atomic diagram of $\mathbf{A}$ iff the mapping $h : \mathbf{A} \to \mathbf{B} : a^{\mathbf{A}} \to a^{\mathbf{B}}$ is an embedding.

Proof. If $\mathbf{B}_A$ is a model of the atomic diagram of $\mathbf{A}$, then $\mathbf{B}_A$ satisfies all L_A-sentences of the form $\pm(a_i = a_j)$, $\pm(F(a_{i_1}, \ldots, a_{i_n}) = a_j)$, $\pm R(a_{i_1}, \ldots, a_{i_n})$ that are true in $\mathbf{A}$. From its definition, h must preserve the satisfaction of these sentences, hence h is an embedding.
Conversely, if $\mathbf{B}_A$ is not a model of the atomic diagram of $\mathbf{A}$, then there is some $\pm$atomic formula $\alpha(\mathbf{x})$ and some tuple $\mathbf{a} \in A^n$ such that $\mathbf{B}_A \not\models \alpha(\mathbf{a})$ while $\mathbf{A}_A \models \alpha(\mathbf{a})$.

Atomic diagrams

Definition. If $\mathbf{A}$ is an L-structure, then the **atomic diagram** of $\mathbf{A}$ is the L_A-theory axiomatized by the $\pm$atomic sentences that hold in $\mathbf{A}_A$. (Write $\text{Diag}(\mathbf{A})$ for this theory.)

Diagram Lemma. $\mathbf{B}_A$ is a model of the atomic diagram of $\mathbf{A}$ iff the mapping $h : \mathbf{A} \to \mathbf{B} : a^{\mathbf{A}} \to a^{\mathbf{B}}$ is an embedding.

Proof. If $\mathbf{B}_A$ is a model of the atomic diagram of $\mathbf{A}$, then $\mathbf{B}_A$ satisfies all L_A-sentences of the form $\pm(a_i = a_j)$, $\pm(F(a_{i_1}, \ldots, a_{i_n}) = a_j)$, $\pm R(a_{i_1}, \ldots, a_{i_n})$ that are true in $\mathbf{A}$. From its definition, h must preserve the satisfaction of these sentences, hence h is an embedding.
Conversely, if $\mathbf{B}_A$ is not a model of the atomic diagram of $\mathbf{A}$, then there is some $\pm$atomic formula $\alpha(\mathbf{x})$ and some tuple $\mathbf{a} \in A^n$ such that $\mathbf{B}_A \not\models \alpha(\mathbf{a})$ while $\mathbf{A}_A \models \alpha(\mathbf{a})$. In this case h does not preserve the truth of $\alpha(\mathbf{a})$, so it is not an embedding.

Atomic diagrams

Definition. If $\mathbf{A}$ is an L-structure, then the **atomic diagram** of $\mathbf{A}$ is the L_A-theory axiomatized by the $\pm$atomic sentences that hold in $\mathbf{A}_A$. (Write $\text{Diag}(\mathbf{A})$ for this theory.)

Diagram Lemma. $\mathbf{B}_A$ is a model of the atomic diagram of $\mathbf{A}$ iff the mapping $h : \mathbf{A} \to \mathbf{B} : a^{\mathbf{A}} \to a^{\mathbf{B}}$ is an embedding.

Proof. If $\mathbf{B}_A$ is a model of the atomic diagram of $\mathbf{A}$, then $\mathbf{B}_A$ satisfies all L_A-sentences of the form $\pm(a_i = a_j)$, $\pm(F(a_{i_1}, \ldots, a_{i_n}) = a_j)$, $\pm R(a_{i_1}, \ldots, a_{i_n})$ that are true in $\mathbf{A}$. From its definition, h must preserve the satisfaction of these sentences, hence h is an embedding.
Conversely, if $\mathbf{B}_A$ is not a model of the atomic diagram of $\mathbf{A}$, then there is some $\pm$atomic formula $\alpha(\mathbf{x})$ and some tuple $\mathbf{a} \in A^n$ such that $\mathbf{B}_A \not\models \alpha(\mathbf{a})$ while $\mathbf{A}_A \models \alpha(\mathbf{a})$. In this case h does not preserve the truth of $\alpha(\mathbf{a})$, so it is not an embedding. $\square$

A ‘local’ characterization of q.e.

A ‘local’ characterization of q.e.

Theorem.

A ‘local’ characterization of q.e.

Theorem. Let T be an L-theory and $\varphi(\mathbf{x})$ be an L-formula. TFAE.

(1) $T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x}))$ for some quantifier free formula $\alpha(\mathbf{x})$.

A 'local' characterization of q.e.

Theorem. Let T be an L-theory and $\varphi(\mathbf{x})$ be an L-formula. TFAE.

(1) $T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x}))$ for some quantifier free formula $\alpha(\mathbf{x})$.

(2) For all $\mathbf{B}, \mathbf{C} \models T$, $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$ and $\mathbf{a} \in A^n$, $\mathbf{B} \models \varphi[\mathbf{a}]$ iff $\mathbf{C} \models \varphi[\mathbf{a}]$.

A ‘local’ characterization of q.e.

Theorem. Let T be an L-theory and $\varphi(\mathbf{x})$ be an L-formula. TFAE.

(1) $T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x}))$ for some quantifier free formula $\alpha(\mathbf{x})$.

(2) For all $\mathbf{B}, \mathbf{C} \models T$, $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$ and $\mathbf{a} \in A^n$, $\mathbf{B} \models \varphi[\mathbf{a}]$ iff $\mathbf{C} \models \varphi[\mathbf{a}]$.

A ‘local’ characterization of q.e.

Theorem. Let T be an L-theory and $\varphi(\mathbf{x})$ be an L-formula. TFAE.

(1) $T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x}))$ for some quantifier free formula $\alpha(\mathbf{x})$.

(2) For all $\mathbf{B}, \mathbf{C} \models T$, $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$ and $\mathbf{a} \in A^n$, $\mathbf{B} \models \varphi[\mathbf{a}]$ iff $\mathbf{C} \models \varphi[\mathbf{a}]$.

Proof.

A ‘local’ characterization of q.e.

Theorem. Let T be an L-theory and $\varphi(\mathbf{x})$ be an L-formula. TFAE.

(1) $T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x}))$ for some quantifier free formula $\alpha(\mathbf{x})$.

(2) For all $\mathbf{B}, \mathbf{C} \models T$, $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$ and $\mathbf{a} \in A^n$, $\mathbf{B} \models \varphi[\mathbf{a}]$ iff $\mathbf{C} \models \varphi[\mathbf{a}]$.

Proof.
[(1)⇒(2)]

A ‘local’ characterization of q.e.

Theorem. Let T be an L-theory and $\varphi(\mathbf{x})$ be an L-formula. TFAE.

(1) $T \models (\forall \mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x}))$ for some quantifier free formula $\alpha(\mathbf{x})$.

(2) For all $\mathbf{B}, \mathbf{C} \models T$, $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$ and $\mathbf{a} \in A^n$, $\mathbf{B} \models \varphi[\mathbf{a}]$ iff $\mathbf{C} \models \varphi[\mathbf{a}]$.

Proof.
[(1)⇒(2)] $\mathbf{B} \models \varphi[\mathbf{a}]$ iff $\mathbf{B} \models \alpha[\mathbf{a}]$ iff $\mathbf{A} \models \alpha[\mathbf{a}]$ iff $\mathbf{C} \models \alpha[\mathbf{a}]$ iff $\mathbf{C} \models \varphi[\mathbf{a}]$.
This uses the fact that q.f. formulas are preserved in passing to substructures or extensions.

A ‘local’ characterization of q.e.

A ‘local’ characterization of q.e.

[(2)$\Rightarrow$(1)]

A 'local' characterization of q.e.

[(2)⇒(1)] Add a tuple $\mathbf{c}$ of new constants to the language with $|\mathbf{c}| = |\mathbf{x}|$. Let

$$\Sigma = \{\alpha(\mathbf{c}) \in L^{\text{q.f.}}(\mathbf{c}) \mid T \models \varphi(\mathbf{c}) \to \alpha(\mathbf{c})\}$$

be the set of q.f. *sentences* in $L_{\mathbf{c}}$ that are consequences of $T \cup \{\varphi(\mathbf{c})\}$.

A ‘local’ characterization of q.e.

[(2)⇒(1)] Add a tuple $\mathbf{c}$ of new constants to the language with $|\mathbf{c}| = |\mathbf{x}|$. Let

$$\Sigma = \{\alpha(\mathbf{c}) \in L^{\text{q.f.}}(\mathbf{c}) \mid T \models \varphi(\mathbf{c}) \to \alpha(\mathbf{c})\}$$

be the set of q.f. *sentences* in $L_{\mathbf{c}}$ that are consequences of $T \cup \{\varphi(\mathbf{c})\}$. (Note that a q.f. $L_{\mathbf{c}}$-sentence $\alpha(\mathbf{c})$ is obtained from a q.f. L-formula $\alpha(\mathbf{x})$ by replacing $\mathbf{x}$ with $\mathbf{c}$.)

A ‘local’ characterization of q.e.

Claim 1.

A ‘local’ characterization of q.e.

Claim 1. $T \cup \Sigma \models \varphi(\mathbf{c})$.

A 'local' characterization of q.e.

Claim 1. $T \cup \Sigma \models \varphi(\mathbf{c})$.

We are working in the language $L_{\mathbf{c}}$.

Claim 1. $T \cup \Sigma \models \varphi(\mathbf{c})$.

We are working in the language $L_{\mathbf{c}}$.
If Claim 1 is false, there is an $L_{\mathbf{c}}$-structure $\mathbf{B_c}$ such that
$\mathbf{B_c} \models T \cup \Sigma \cup \{\neg\varphi(\mathbf{c})\}$.

Claim 1. $T \cup \Sigma \models \varphi(\mathbf{c})$.

We are working in the language $L_{\mathbf{c}}$.
If Claim 1 is false, there is an $L_{\mathbf{c}}$-structure $\mathbf{B}_{\mathbf{c}}$ such that $\mathbf{B}_{\mathbf{c}} \models T \cup \Sigma \cup \{\neg\varphi(\mathbf{c})\}$. Let $\mathbf{A}_{\mathbf{c}} = \langle \mathbf{c} \rangle_{\mathbf{B}_{\mathbf{c}}}$ be the substructure in $\mathbf{B}_{\mathbf{c}}$ generated by (the elements in) $\mathbf{c}$. For the underlying L-structures $\mathbf{A}$ and $\mathbf{B}$ we have $\mathbf{B} \models T$, $\mathbf{A} \leq \mathbf{B}$, $\mathbf{c} \in A$ and $\mathbf{B} \not\models \varphi[\mathbf{c}]$. To conclude the proof of the claim we intend to obtain a contradiction using condition (2) of the theorem.

A ‘local’ characterization of q.e.

Claim 1. $T \cup \Sigma \models \varphi(\mathbf{c})$.

We are working in the language $L_{\mathbf{c}}$.
If Claim 1 is false, there is an $L_{\mathbf{c}}$-structure $\mathbf{B}_{\mathbf{c}}$ such that $\mathbf{B}_{\mathbf{c}} \models T \cup \Sigma \cup \{\neg\varphi(\mathbf{c})\}$. Let $\mathbf{A}_{\mathbf{c}} = \langle \mathbf{c} \rangle_{\mathbf{B}_{\mathbf{c}}}$ be the substructure in $\mathbf{B}_{\mathbf{c}}$ generated by (the elements in) $\mathbf{c}$. For the underlying L-structures $\mathbf{A}$ and $\mathbf{B}$ we have $\mathbf{B} \models T$, $\mathbf{A} \leq \mathbf{B}$, $\mathbf{c} \in A$ and $\mathbf{B} \not\models \varphi[\mathbf{c}]$. To conclude the proof of the claim we intend to obtain a contradiction using condition (2) of the theorem. To do this it suffices to exhibit an L-structure $\mathbf{C}$ such that $\mathbf{C} \models T$, $\mathbf{A} \leq \mathbf{C}$ and $\mathbf{C} \models \varphi[\mathbf{c}]$.

Claim 1. $T \cup \Sigma \models \varphi(\mathbf{c})$.

We are working in the language $L_{\mathbf{c}}$.
If Claim 1 is false, there is an $L_{\mathbf{c}}$-structure $\mathbf{B_c}$ such that $\mathbf{B_c} \models T \cup \Sigma \cup \{\neg\varphi(\mathbf{c})\}$. Let $\mathbf{A_c} = \langle \mathbf{c} \rangle_{\mathbf{B_c}}$ be the substructure in $\mathbf{B_c}$ generated by (the elements in) $\mathbf{c}$. For the underlying L-structures $\mathbf{A}$ and $\mathbf{B}$ we have $\mathbf{B} \models T$, $\mathbf{A} \leq \mathbf{B}$, $\mathbf{c} \in A$ and $\mathbf{B} \not\models \varphi[\mathbf{c}]$. To conclude the proof of the claim we intend to obtain a contradiction using condition (2) of the theorem. To do this it suffices to exhibit an L-structure $\mathbf{C}$ such that $\mathbf{C} \models T$, $\mathbf{A} \leq \mathbf{C}$ and $\mathbf{C} \models \varphi[\mathbf{c}]$. Therefore let $\Gamma = T \cup \text{Diag}(\mathbf{A_c}) \cup \{\varphi(\mathbf{c})\}$.

A ‘local’ characterization of q.e.

Subclaim 2.

A ‘local’ characterization of q.e.

Subclaim 2. Γ has a model.

A ‘local’ characterization of q.e.

Subclaim 2. Γ has a model.

If not, then there is a finite conjunction $\psi(\mathbf{c})$ of members of $\mathrm{Diag}(\mathbf{A_c})$ such that $T \cup \{\psi(\mathbf{c}), \varphi(\mathbf{c})\}$ has no model.

A ‘local’ characterization of q.e.

Subclaim 2. Γ has a model.

If not, then there is a finite conjunction $\psi(\mathbf{c})$ of members of $\mathrm{Diag}(\mathbf{A_c})$ such that $T \cup \{\psi(\mathbf{c}), \varphi(\mathbf{c})\}$ has no model. This fact is expressible as $T \models \varphi(\mathbf{c}) \to \neg\psi(\mathbf{c})$, hence $\neg\psi(\mathbf{c}) \in \Sigma$.

A ‘local’ characterization of q.e.

Subclaim 2. Γ has a model.

If not, then there is a finite conjunction $\psi(\mathbf{c})$ of members of $\text{Diag}(\mathbf{A_c})$ such that $T \cup \{\psi(\mathbf{c}), \varphi(\mathbf{c})\}$ has no model. This fact is expressible as $T \models \varphi(\mathbf{c}) \to \neg\psi(\mathbf{c})$, hence $\neg\psi(\mathbf{c}) \in \Sigma$. Hence $\mathbf{B_c} \models \neg\psi(\mathbf{c})$, hence $\mathbf{A_c} \models \neg\psi(\mathbf{c})$ since $\psi(\mathbf{c})$ is q.f.

A ‘local’ characterization of q.e.

Subclaim 2. Γ has a model.

If not, then there is a finite conjunction $\psi(\mathbf{c})$ of members of $\mathrm{Diag}(\mathbf{A_c})$ such that $T \cup \{\psi(\mathbf{c}), \varphi(\mathbf{c})\}$ has no model. This fact is expressible as $T \models \varphi(\mathbf{c}) \to \neg\psi(\mathbf{c})$, hence $\neg\psi(\mathbf{c}) \in \Sigma$. Hence $\mathbf{B_c} \models \neg\psi(\mathbf{c})$, hence $\mathbf{A_c} \models \neg\psi(\mathbf{c})$ since $\psi(\mathbf{c})$ is q.f. But $\psi(\mathbf{c}) \in \mathrm{Diag}(\mathbf{A_c})$, so this is a contradiction.

A ‘local’ characterization of q.e.

Subclaim 2. Γ has a model.

If not, then there is a finite conjunction $\psi(\mathbf{c})$ of members of $\mathrm{Diag}(\mathbf{A_c})$ such that $T \cup \{\psi(\mathbf{c}), \varphi(\mathbf{c})\}$ has no model. This fact is expressible as $T \models \varphi(\mathbf{c}) \to \neg\psi(\mathbf{c})$, hence $\neg\psi(\mathbf{c}) \in \Sigma$. Hence $\mathbf{B_c} \models \neg\psi(\mathbf{c})$, hence $\mathbf{A_c} \models \neg\psi(\mathbf{c})$ since $\psi(\mathbf{c})$ is q.f. But $\psi(\mathbf{c}) \in \mathrm{Diag}(\mathbf{A_c})$, so this is a contradiction. The Subclaim is proved.

A ‘local’ characterization of q.e.

Subclaim 2. Γ has a model.

If not, then there is a finite conjunction $\psi(\mathbf{c})$ of members of $\mathrm{Diag}(\mathbf{A_c})$ such that $T \cup \{\psi(\mathbf{c}), \varphi(\mathbf{c})\}$ has no model. This fact is expressible as $T \models \varphi(\mathbf{c}) \to \neg\psi(\mathbf{c})$, hence $\neg\psi(\mathbf{c}) \in \Sigma$. Hence $\mathbf{B_c} \models \neg\psi(\mathbf{c})$, hence $\mathbf{A_c} \models \neg\psi(\mathbf{c})$ since $\psi(\mathbf{c})$ is q.f. But $\psi(\mathbf{c}) \in \mathrm{Diag}(\mathbf{A_c})$, so this is a contradiction. The Subclaim is proved.

The Subclaim and condition (2) of the theorem lead to a contradiction in the proof of the Claim.

Subclaim 2. Γ has a model.

If not, then there is a finite conjunction $\psi(\mathbf{c})$ of members of $\mathrm{Diag}(\mathbf{A_c})$ such that $T \cup \{\psi(\mathbf{c}), \varphi(\mathbf{c})\}$ has no model. This fact is expressible as $T \models \varphi(\mathbf{c}) \to \neg\psi(\mathbf{c})$, hence $\neg\psi(\mathbf{c}) \in \Sigma$. Hence $\mathbf{B_c} \models \neg\psi(\mathbf{c})$, hence $\mathbf{A_c} \models \neg\psi(\mathbf{c})$ since $\psi(\mathbf{c})$ is q.f. But $\psi(\mathbf{c}) \in \mathrm{Diag}(\mathbf{A_c})$, so this is a contradiction. The Subclaim is proved.

The Subclaim and condition (2) of the theorem lead to a contradiction in the proof of the Claim. Namely, if Γ has a model, $\mathbf{C_c}$, then the $\mathcal{L}$-reducts of $\mathbf{A_c}, \mathbf{B_c}$, and $\mathbf{C_c}$ are structures satisfying $\mathbf{B}, \mathbf{C} \models T$, $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$, $\mathbf{a} = \mathbf{c^A}$, $\mathbf{B} \models \neg\varphi[\mathbf{a}]$ while $\mathbf{C} \models \varphi[\mathbf{a}]$.

A ‘local’ characterization of q.e.

Subclaim 2. Γ has a model.

If not, then there is a finite conjunction $\psi(\mathbf{c})$ of members of $\mathrm{Diag}(\mathbf{A_c})$ such that $T \cup \{\psi(\mathbf{c}), \varphi(\mathbf{c})\}$ has no model. This fact is expressible as $T \models \varphi(\mathbf{c}) \to \neg\psi(\mathbf{c})$, hence $\neg\psi(\mathbf{c}) \in \Sigma$. Hence $\mathbf{B_c} \models \neg\psi(\mathbf{c})$, hence $\mathbf{A_c} \models \neg\psi(\mathbf{c})$ since $\psi(\mathbf{c})$ is q.f. But $\psi(\mathbf{c}) \in \mathrm{Diag}(\mathbf{A_c})$, so this is a contradiction. The Subclaim is proved.

The Subclaim and condition (2) of the theorem lead to a contradiction in the proof of the Claim. Namely, if Γ has a model, $\mathbf{C_c}$, then the $\mathcal{L}$-reducts of $\mathbf{A_c}, \mathbf{B_c}$, and $\mathbf{C_c}$ are structures satisfying $\mathbf{B}, \mathbf{C} \models T$, $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$, $\mathbf{a} = \mathbf{c}^{\mathbf{A}}$, $\mathbf{B} \models \neg\varphi[\mathbf{a}]$ while $\mathbf{C} \models \varphi[\mathbf{a}]$. The Claim is proved.

Subclaim 2. Γ has a model.

If not, then there is a finite conjunction $\psi(\mathbf{c})$ of members of Diag$(\mathbf{A}_\mathbf{c})$ such that $T \cup \{\psi(\mathbf{c}), \varphi(\mathbf{c})\}$ has no model. This fact is expressible as $T \models \varphi(\mathbf{c}) \to \neg\psi(\mathbf{c})$, hence $\neg\psi(\mathbf{c}) \in \Sigma$. Hence $\mathbf{B}_\mathbf{c} \models \neg\psi(\mathbf{c})$, hence $\mathbf{A}_\mathbf{c} \models \neg\psi(\mathbf{c})$ since $\psi(\mathbf{c})$ is q.f. But $\psi(\mathbf{c}) \in$ Diag$(\mathbf{A}_\mathbf{c})$, so this is a contradiction. The Subclaim is proved.

The Subclaim and condition (2) of the theorem lead to a contradiction in the proof of the Claim. Namely, if Γ has a model, $\mathbf{C}_\mathbf{c}$, then the $\mathcal{L}$-reducts of $\mathbf{A}_\mathbf{c}, \mathbf{B}_\mathbf{c}$, and $\mathbf{C}_\mathbf{c}$ are structures satisfying $\mathbf{B}, \mathbf{C} \models T$, $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$, $\mathbf{a} = \mathbf{c}^\mathbf{A}$, $\mathbf{B} \models \neg\varphi[\mathbf{a}]$ while $\mathbf{C} \models \varphi[\mathbf{a}]$. The Claim is proved.

To complete the proof of the theorem, let $\alpha(\mathbf{c}) = \wedge\alpha_i(\mathbf{c})$ be a finite conjunction of members of Σ for which $T \cup \{\alpha(\mathbf{c})\} \models \varphi(\mathbf{c})$;

Subclaim 2. Γ has a model.

If not, then there is a finite conjunction $\psi(\mathbf{c})$ of members of $\mathrm{Diag}(\mathbf{A_c})$ such that $T \cup \{\psi(\mathbf{c}), \varphi(\mathbf{c})\}$ has no model. This fact is expressible as $T \models \varphi(\mathbf{c}) \to \neg\psi(\mathbf{c})$, hence $\neg\psi(\mathbf{c}) \in \Sigma$. Hence $\mathbf{B_c} \models \neg\psi(\mathbf{c})$, hence $\mathbf{A_c} \models \neg\psi(\mathbf{c})$ since $\psi(\mathbf{c})$ is q.f. But $\psi(\mathbf{c}) \in \mathrm{Diag}(\mathbf{A_c})$, so this is a contradiction. The Subclaim is proved.

The Subclaim and condition (2) of the theorem lead to a contradiction in the proof of the Claim. Namely, if Γ has a model, $\mathbf{C_c}$, then the $\mathcal{L}$-reducts of $\mathbf{A_c}, \mathbf{B_c}$, and $\mathbf{C_c}$ are structures satisfying $\mathbf{B}, \mathbf{C} \models T$, $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$, $\mathbf{a} = \mathbf{c}^{\mathbf{A}}$, $\mathbf{B} \models \neg\varphi[\mathbf{a}]$ while $\mathbf{C} \models \varphi[\mathbf{a}]$. The Claim is proved.

To complete the proof of the theorem, let $\alpha(\mathbf{c}) = \wedge\alpha_i(\mathbf{c})$ be a finite conjunction of members of Σ for which $T \cup \{\alpha(\mathbf{c})\} \models \varphi(\mathbf{c})$; i.e. $T \models \alpha(\mathbf{c}) \to \varphi(\mathbf{c})$.

Subclaim 2. Γ has a model.

If not, then there is a finite conjunction $\psi(\mathbf{c})$ of members of $\text{Diag}(\mathbf{A_c})$ such that $T \cup \{\psi(\mathbf{c}), \varphi(\mathbf{c})\}$ has no model. This fact is expressible as $T \models \varphi(\mathbf{c}) \to \neg\psi(\mathbf{c})$, hence $\neg\psi(\mathbf{c}) \in \Sigma$. Hence $\mathbf{B_c} \models \neg\psi(\mathbf{c})$, hence $\mathbf{A_c} \models \neg\psi(\mathbf{c})$ since $\psi(\mathbf{c})$ is q.f. But $\psi(\mathbf{c}) \in \text{Diag}(\mathbf{A_c})$, so this is a contradiction. The Subclaim is proved.

The Subclaim and condition (2) of the theorem lead to a contradiction in the proof of the Claim. Namely, if Γ has a model, $\mathbf{C_c}$, then the $\mathcal{L}$-reducts of $\mathbf{A_c}, \mathbf{B_c}$, and $\mathbf{C_c}$ are structures satisfying $\mathbf{B}, \mathbf{C} \models T$, $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$, $\mathbf{a} = \mathbf{c}^{\mathbf{A}}$, $\mathbf{B} \models \neg\varphi[\mathbf{a}]$ while $\mathbf{C} \models \varphi[\mathbf{a}]$. The Claim is proved.

To complete the proof of the theorem, let $\alpha(\mathbf{c}) = \wedge\alpha_i(\mathbf{c})$ be a finite conjunction of members of Σ for which $T \cup \{\alpha(\mathbf{c})\} \models \varphi(\mathbf{c})$; i.e. $T \models \alpha(\mathbf{c}) \to \varphi(\mathbf{c})$. Since $\alpha(\mathbf{c}) = \wedge\alpha_i(\mathbf{c})$ and $T \models \varphi(\mathbf{c}) \to \alpha_i(\mathbf{c})$ for each i, we have $T \models \varphi(\mathbf{c}) \leftrightarrow \alpha(\mathbf{c})$. Since $\mathbf{c}$ does not appear in T,

Subclaim 2. Γ has a model.

If not, then there is a finite conjunction $\psi(\mathbf{c})$ of members of Diag($\mathbf{A_c}$) such that $T \cup \{\psi(\mathbf{c}), \varphi(\mathbf{c})\}$ has no model. This fact is expressible as $T \models \varphi(\mathbf{c}) \to \neg\psi(\mathbf{c})$, hence $\neg\psi(\mathbf{c}) \in \Sigma$. Hence $\mathbf{B_c} \models \neg\psi(\mathbf{c})$, hence $\mathbf{A_c} \models \neg\psi(\mathbf{c})$ since $\psi(\mathbf{c})$ is q.f. But $\psi(\mathbf{c}) \in$ Diag($\mathbf{A_c}$), so this is a contradiction. The Subclaim is proved.

The Subclaim and condition (2) of the theorem lead to a contradiction in the proof of the Claim. Namely, if Γ has a model, $\mathbf{C_c}$, then the $\mathcal{L}$-reducts of $\mathbf{A_c}, \mathbf{B_c}$, and $\mathbf{C_c}$ are structures satisfying $\mathbf{B}, \mathbf{C} \models T$, $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$, $\mathbf{a} = \mathbf{c}^{\mathbf{A}}$, $\mathbf{B} \models \neg\varphi[\mathbf{a}]$ while $\mathbf{C} \models \varphi[\mathbf{a}]$. The Claim is proved.

To complete the proof of the theorem, let $\alpha(\mathbf{c}) = \wedge\alpha_i(\mathbf{c})$ be a finite conjunction of members of Σ for which $T \cup \{\alpha(\mathbf{c})\} \models \varphi(\mathbf{c})$; i.e. $T \models \alpha(\mathbf{c}) \to \varphi(\mathbf{c})$. Since $\alpha(\mathbf{c}) = \wedge\alpha_i(\mathbf{c})$ and $T \models \varphi(\mathbf{c}) \to \alpha_i(\mathbf{c})$ for each i, we have $T \models \varphi(\mathbf{c}) \leftrightarrow \alpha(\mathbf{c})$. Since $\mathbf{c}$ does not appear in T, $T \models (\forall\mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x}))$.

Subclaim 2. Γ has a model.

If not, then there is a finite conjunction $\psi(\mathbf{c})$ of members of Diag$(\mathbf{A_c})$ such that $T \cup \{\psi(\mathbf{c}), \varphi(\mathbf{c})\}$ has no model. This fact is expressible as $T \models \varphi(\mathbf{c}) \to \neg\psi(\mathbf{c})$, hence $\neg\psi(\mathbf{c}) \in \Sigma$. Hence $\mathbf{B_c} \models \neg\psi(\mathbf{c})$, hence $\mathbf{A_c} \models \neg\psi(\mathbf{c})$ since $\psi(\mathbf{c})$ is q.f. But $\psi(\mathbf{c}) \in$ Diag$(\mathbf{A_c})$, so this is a contradiction. The Subclaim is proved.

The Subclaim and condition (2) of the theorem lead to a contradiction in the proof of the Claim. Namely, if Γ has a model, $\mathbf{C_c}$, then the $\mathcal{L}$-reducts of $\mathbf{A_c}, \mathbf{B_c}$, and $\mathbf{C_c}$ are structures satisfying $\mathbf{B}, \mathbf{C} \models T$, $\mathbf{A} \leq \mathbf{B}, \mathbf{C}$, $\mathbf{a} = \mathbf{c}^{\mathbf{A}}$, $\mathbf{B} \models \neg\varphi[\mathbf{a}]$ while $\mathbf{C} \models \varphi[\mathbf{a}]$. The Claim is proved.

To complete the proof of the theorem, let $\alpha(\mathbf{c}) = \wedge\alpha_i(\mathbf{c})$ be a finite conjunction of members of Σ for which $T \cup \{\alpha(\mathbf{c})\} \models \varphi(\mathbf{c})$; i.e. $T \models \alpha(\mathbf{c}) \to \varphi(\mathbf{c})$. Since $\alpha(\mathbf{c}) = \wedge\alpha_i(\mathbf{c})$ and $T \models \varphi(\mathbf{c}) \to \alpha_i(\mathbf{c})$ for each i, we have $T \models \varphi(\mathbf{c}) \leftrightarrow \alpha(\mathbf{c})$. Since $\mathbf{c}$ does not appear in T, $T \models (\forall\mathbf{x})(\varphi(\mathbf{x}) \leftrightarrow \alpha(\mathbf{x}))$. $\square$

A 'local' characterization of q.e.

A ‘local’ characterization of q.e.

Corollary.

A ‘local’ characterization of q.e.

Corollary. T has q.e. if and only if for all $\mathbf{B}, \mathbf{C} \models T$,

A 'local' characterization of q.e.

Corollary. T has q.e. if and only if for all $\mathbf{B}, \mathbf{C} \models T$, if $\mathbf{b} \in \mathbf{B}^n$ and $\mathbf{c} \in \mathbf{C}^n$,

A ‘local’ characterization of q.e.

Corollary. T has q.e. if and only if for all $\mathbf{B}, \mathbf{C} \models T$, if $\mathbf{b} \in \mathbf{B}^n$ and $\mathbf{c} \in \mathbf{C}^n$, $\mathrm{typ}^{q.f.}_{\mathbf{B}}(\mathbf{b}) = \mathrm{typ}^{q.f.}_{\mathbf{C}}(\mathbf{c})$ implies $\mathrm{typ}_{\mathbf{B}}(\mathbf{b}) = \mathrm{typ}_{\mathbf{C}}(\mathbf{c})$.

A ‘local’ characterization of q.e.

Corollary. T has q.e. if and only if for all $\mathbf{B}, \mathbf{C} \models T$, if $\mathbf{b} \in \mathbf{B}^n$ and $\mathbf{c} \in \mathbf{C}^n$, $\text{typ}_{\mathbf{B}}^{q.f.}(\mathbf{b}) = \text{typ}_{\mathbf{C}}^{q.f.}(\mathbf{c})$ implies $\text{typ}_{\mathbf{B}}(\mathbf{b}) = \text{typ}_{\mathbf{C}}(\mathbf{c})$.

Corollary.

A 'local' characterization of q.e.

Corollary. T has q.e. if and only if for all $\mathbf{B}, \mathbf{C} \models T$, if $\mathbf{b} \in \mathbf{B}^n$ and $\mathbf{c} \in \mathbf{C}^n$, $\text{typ}_{\mathbf{B}}^{q.f.}(\mathbf{b}) = \text{typ}_{\mathbf{C}}^{q.f.}(\mathbf{c})$ implies $\text{typ}_{\mathbf{B}}(\mathbf{b}) = \text{typ}_{\mathbf{C}}(\mathbf{c})$.

Corollary. An L-theory T has q.e. if and only if, whenever $\mathbf{A}$ is a substructure of a model of T, the L_A-theory $T \cup \text{Diag}(\mathbf{A})$ is complete.

A 'local' characterization of q.e.

Corollary. T has q.e. if and only if for all $\mathbf{B}, \mathbf{C} \models T$, if $\mathbf{b} \in \mathbf{B}^n$ and $\mathbf{c} \in \mathbf{C}^n$, $\text{typ}^{q.f.}_{\mathbf{B}}(\mathbf{b}) = \text{typ}^{q.f.}_{\mathbf{C}}(\mathbf{c})$ implies $\text{typ}_{\mathbf{B}}(\mathbf{b}) = \text{typ}_{\mathbf{C}}(\mathbf{c})$.

Corollary. An L-theory T has q.e. if and only if, whenever $\mathbf{A}$ is a substructure of a model of T, the L_A-theory $T \cup \text{Diag}(\mathbf{A})$ is complete.