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5. becomes easier to classify definable sets in models.

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This is of theoretical value, and typically doesn't bypass any practical complications.

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"Arrangement" of variables: For some equivalence relation E on $\{1, \ldots, n\}$

$$\mathrm{Arr}_E(x_1, \ldots, x_n) = \bigwedge_{(i,j)\in E} (x_i = x_j) \wedge \bigwedge_{(i,j)\notin E} \neg(x_i = x_j).$$

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This is a conjunction of $\pm$atomic.

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Hence, if $\varphi(\mathbf{x}, y)$ is q.f., $T \models (\forall \mathbf{x})((\exists y)\varphi(\mathbf{x}, y) \leftrightarrow (\exists y)(\bigvee_k \mathrm{Arr}_{E_k}(\mathbf{x}, y)))$.

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A brute force argument that DLO has q.e. can be modeled on this one.