The category of L-structures

Functions = morphisms of sets

Functions = morphisms of sets

In mathematics, we typically compare structures of the same type with functions:

Functions = morphisms of sets

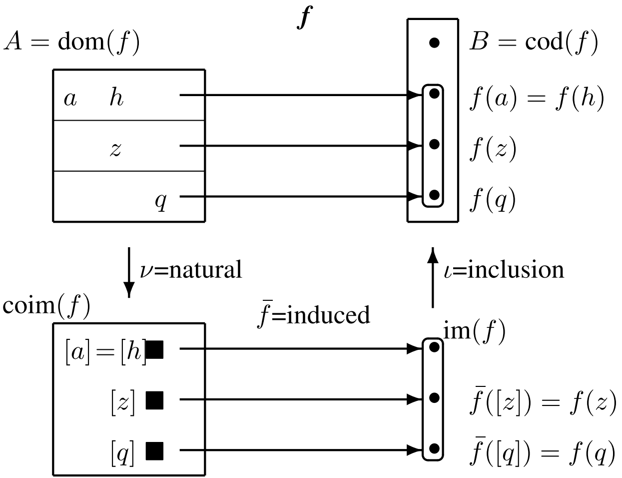
In mathematics, we typically compare structures of the same type with functions: $f\colon A \to B$.

Functions = morphisms of sets

In mathematics, we typically compare structures of the same type with functions: $f\colon A \to B$. Starting with a set-function, there are derived concepts:

Functions = morphisms of sets

In mathematics, we typically compare structures of the same type with functions: $f\colon A \to B$. Starting with a set-function, there are derived concepts:



Concepts derived from “morphism”

Concepts derived from "morphism"

1. image, coimage

Concepts derived from “morphism”

1. image, coimage

Concepts derived from “morphism”

1. image, coimage
2. substructure, quotient

Concepts derived from "morphism"

1. image, coimage
2. substructure, quotient

Concepts derived from “morphism”

1. image, coimage
2. substructure, quotient
3. natural map, induced map, inclusion map

Concepts derived from “morphism”

1. image, coimage
2. substructure, quotient
3. natural map, induced map, inclusion map

Concepts derived from “morphism”

1. image, coimage
2. substructure, quotient
3. natural map, induced map, inclusion map
4. embedding, isomorphism

Concepts derived from “morphism”

1. image, coimage
2. substructure, quotient
3. natural map, induced map, inclusion map
4. embedding, isomorphism

Concepts derived from “morphism”

1. image, coimage
2. substructure, quotient
3. natural map, induced map, inclusion map
4. embedding, isomorphism
5. product, coproduct

Concepts derived from “morphism”

1. image, coimage
2. substructure, quotient
3. natural map, induced map, inclusion map
4. embedding, isomorphism
5. product, coproduct

Concepts derived from “morphism”

1. image, coimage
2. substructure, quotient
3. natural map, induced map, inclusion map
4. embedding, isomorphism
5. product, coproduct

Morphisms of first-order structures

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature,

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$,

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$,

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$, is a function $h\colon A \to B$ between their universes which **preserves** the structure in the sense that

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$, is a function $h\colon A \to B$ between their universes which **preserves** the structure in the sense that

1. (Constants are preserved)

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$, is a function $h\colon A \to B$ between their universes which **preserves** the structure in the sense that

1. (Constants are preserved)

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$, is a function $h\colon A \to B$ between their universes which **preserves** the structure in the sense that

1. (Constants are preserved) $h(c^{\mathbf{A}}) = c^{\mathbf{B}}$ for every constant symbol c.

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$, is a function $h\colon A \to B$ between their universes which **preserves** the structure in the sense that

1. (Constants are preserved) $h(c^{\mathbf{A}}) = c^{\mathbf{B}}$ for every constant symbol c.
2. (Functions/Operations are preserved)

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$, is a function $h\colon A \to B$ between their universes which **preserves** the structure in the sense that

1. (Constants are preserved) $h(c^{\mathbf{A}}) = c^{\mathbf{B}}$ for every constant symbol c.
2. (Functions/Operations are preserved)

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$, is a function $h\colon A \to B$ between their universes which **preserves** the structure in the sense that

1. (Constants are preserved) $h(c^{\mathbf{A}}) = c^{\mathbf{B}}$ for every constant symbol c.
2. (Functions/Operations are preserved) $h(F^{\mathbf{A}}(a_1, \ldots, a_n)) = F^{\mathbf{B}}(h(a_1), \ldots, h(a_n))$ for every operation symbol F.

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$, is a function $h\colon A \to B$ between their universes which **preserves** the structure in the sense that

1. (Constants are preserved) $h(c^{\mathbf{A}}) = c^{\mathbf{B}}$ for every constant symbol c.
2. (Functions/Operations are preserved) $h(F^{\mathbf{A}}(a_1, \ldots, a_n)) = F^{\mathbf{B}}(h(a_1), \ldots, h(a_n))$ for every operation symbol F.
3. (Relations/Predicates are preserved)

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$, is a function $h\colon A \to B$ between their universes which **preserves** the structure in the sense that

1. (Constants are preserved) $h(c^{\mathbf{A}}) = c^{\mathbf{B}}$ for every constant symbol c.
2. (Functions/Operations are preserved) $h(F^{\mathbf{A}}(a_1, \ldots, a_n)) = F^{\mathbf{B}}(h(a_1), \ldots, h(a_n))$ for every operation symbol F.
3. (Relations/Predicates are preserved)

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$, is a function $h\colon A \to B$ between their universes which **preserves** the structure in the sense that

1. (Constants are preserved) $h(c^{\mathbf{A}}) = c^{\mathbf{B}}$ for every constant symbol c.
2. (Functions/Operations are preserved) $h(F^{\mathbf{A}}(a_1, \ldots, a_n)) = F^{\mathbf{B}}(h(a_1), \ldots, h(a_n))$ for every operation symbol F.
3. (Relations/Predicates are preserved) $R^{\mathbf{A}}(a_1, \ldots, a_n) = \top \implies R^{\mathbf{B}}(h(a_1), \ldots, h(a_n)) = \top$ for every predicate symbol R.

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$, is a function $h\colon A \to B$ between their universes which **preserves** the structure in the sense that

1. (Constants are preserved) $h(c^{\mathbf{A}}) = c^{\mathbf{B}}$ for every constant symbol c.
2. (Functions/Operations are preserved) $h(F^{\mathbf{A}}(a_1, \ldots, a_n)) = F^{\mathbf{B}}(h(a_1), \ldots, h(a_n))$ for every operation symbol F.
3. (Relations/Predicates are preserved) $R^{\mathbf{A}}(a_1, \ldots, a_n) = \top \implies R^{\mathbf{B}}(h(a_1), \ldots, h(a_n)) = \top$ for every predicate symbol R.

Definition. An **isomorphism** is an invertible homomorphism.

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$, is a function $h\colon A \to B$ between their universes which **preserves** the structure in the sense that

1. (Constants are preserved) $h(c^{\mathbf{A}}) = c^{\mathbf{B}}$ for every constant symbol c.
2. (Functions/Operations are preserved) $h(F^{\mathbf{A}}(a_1, \ldots, a_n)) = F^{\mathbf{B}}(h(a_1), \ldots, h(a_n))$ for every operation symbol F.
3. (Relations/Predicates are preserved) $R^{\mathbf{A}}(a_1, \ldots, a_n) = \top \implies R^{\mathbf{B}}(h(a_1), \ldots, h(a_n)) = \top$ for every predicate symbol R.

Definition. An **isomorphism** is an invertible homomorphism.
(It is a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$, is a function $h\colon A \to B$ between their universes which **preserves** the structure in the sense that

1. (Constants are preserved) $h(c^{\mathbf{A}}) = c^{\mathbf{B}}$ for every constant symbol c.
2. (Functions/Operations are preserved) $h(F^{\mathbf{A}}(a_1, \ldots, a_n)) = F^{\mathbf{B}}(h(a_1), \ldots, h(a_n))$ for every operation symbol F.
3. (Relations/Predicates are preserved) $R^{\mathbf{A}}(a_1, \ldots, a_n) = \top \implies R^{\mathbf{B}}(h(a_1), \ldots, h(a_n)) = \top$ for every predicate symbol R.

Definition. An **isomorphism** is an invertible homomorphism.
(It is a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ which is invertible as a set-function,

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$, is a function $h\colon A \to B$ between their universes which **preserves** the structure in the sense that

1. (Constants are preserved) $h(c^{\mathbf{A}}) = c^{\mathbf{B}}$ for every constant symbol c.
2. (Functions/Operations are preserved) $h(F^{\mathbf{A}}(a_1, \ldots, a_n)) = F^{\mathbf{B}}(h(a_1), \ldots, h(a_n))$ for every operation symbol F.
3. (Relations/Predicates are preserved) $R^{\mathbf{A}}(a_1, \ldots, a_n) = \top \implies R^{\mathbf{B}}(h(a_1), \ldots, h(a_n)) = \top$ for every predicate symbol R.

Definition. An **isomorphism** is an invertible homomorphism.
(It is a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ which is invertible as a set-function, and which has the property that $h^{-1}\colon \mathbf{B} \to \mathbf{A}$ is also a homomorphism.)

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$, is a function $h\colon A \to B$ between their universes which **preserves** the structure in the sense that

1. (Constants are preserved) $h(c^{\mathbf{A}}) = c^{\mathbf{B}}$ for every constant symbol c.
2. (Functions/Operations are preserved) $h(F^{\mathbf{A}}(a_1, \ldots, a_n)) = F^{\mathbf{B}}(h(a_1), \ldots, h(a_n))$ for every operation symbol F.
3. (Relations/Predicates are preserved) $R^{\mathbf{A}}(a_1, \ldots, a_n) = \top \Longrightarrow R^{\mathbf{B}}(h(a_1), \ldots, h(a_n)) = \top$ for every predicate symbol R.

Definition. An **isomorphism** is an invertible homomorphism. (It is a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ which is invertible as a set-function, and which has the property that $h^{-1}\colon \mathbf{B} \to \mathbf{A}$ is also a homomorphism.) An isomorphism from a structure to itself,

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$, is a function $h\colon A \to B$ between their universes which **preserves** the structure in the sense that

1. (Constants are preserved) $h(c^{\mathbf{A}}) = c^{\mathbf{B}}$ for every constant symbol c.
2. (Functions/Operations are preserved) $h(F^{\mathbf{A}}(a_1, \ldots, a_n)) = F^{\mathbf{B}}(h(a_1), \ldots, h(a_n))$ for every operation symbol F.
3. (Relations/Predicates are preserved) $R^{\mathbf{A}}(a_1, \ldots, a_n) = \top \implies R^{\mathbf{B}}(h(a_1), \ldots, h(a_n)) = \top$ for every predicate symbol R.

Definition. An **isomorphism** is an invertible homomorphism. (It is a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ which is invertible as a set-function, and which has the property that $h^{-1}\colon \mathbf{B} \to \mathbf{A}$ is also a homomorphism.) An isomorphism from a structure to itself, $h\colon \mathbf{A} \to \mathbf{A}$,

Morphisms of first-order structures

Definition. If $\mathbf{A}$ and $\mathbf{B}$ are structures of the same signature, then a **homomorphism** from $\mathbf{A}$ to $\mathbf{B}$, $h\colon \mathbf{A} \to \mathbf{B}$, is a function $h\colon A \to B$ between their universes which **preserves** the structure in the sense that

1. (Constants are preserved) $h(c^{\mathbf{A}}) = c^{\mathbf{B}}$ for every constant symbol c.
2. (Functions/Operations are preserved) $h(F^{\mathbf{A}}(a_1, \ldots, a_n)) = F^{\mathbf{B}}(h(a_1), \ldots, h(a_n))$ for every operation symbol F.
3. (Relations/Predicates are preserved) $R^{\mathbf{A}}(a_1, \ldots, a_n) = \top \implies R^{\mathbf{B}}(h(a_1), \ldots, h(a_n)) = \top$ for every predicate symbol R.

Definition. An **isomorphism** is an invertible homomorphism. (It is a homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ which is invertible as a set-function, and which has the property that $h^{-1}\colon \mathbf{B} \to \mathbf{A}$ is also a homomorphism.) An isomorphism from a structure to itself, $h\colon \mathbf{A} \to \mathbf{A}$, is an **automorphism**.

“Substructure” captures “image”

“Substructure” captures “image”

Definition.

“Substructure” captures “image”

Definition. If $\mathbf{B}$ is a structure, then $\mathbf{S}$ is a **substructure** of $\mathbf{B}$,

“Substructure” captures “image”

Definition. If $\mathbf{B}$ is a structure, then $\mathbf{S}$ is a **substructure** of $\mathbf{B}$, $\mathbf{S} \leq \mathbf{B}$,

“Substructure” captures “image”

Definition. If $\mathbf{B}$ is a structure, then $\mathbf{S}$ is a **substructure** of $\mathbf{B}$, $\mathbf{S} \leq \mathbf{B}$, if

“Substructure” captures “image”

Definition. If $\mathbf{B}$ is a structure, then $\mathbf{S}$ is a **substructure** of $\mathbf{B}$, $\mathbf{S} \leq \mathbf{B}$, if

1. $S \subseteq B$ and

“Substructure” captures “image”

Definition. If $\mathbf{B}$ is a structure, then $\mathbf{S}$ is a **substructure** of $\mathbf{B}$, $\mathbf{S} \leq \mathbf{B}$, if

1. $S \subseteq B$ and

“Substructure” captures “image”

Definition. If $\mathbf{B}$ is a structure, then $\mathbf{S}$ is a **substructure** of $\mathbf{B}$, $\mathbf{S} \leq \mathbf{B}$, if

1. $S \subseteq B$ and
2. the inclusion map $\iota \colon S \to B$ preserves **and reflects** the constants, operations, and the predicates.

“Substructure” captures “image”

Definition. If $\mathbf{B}$ is a structure, then $\mathbf{S}$ is a **substructure** of $\mathbf{B}$, $\mathbf{S} \leq \mathbf{B}$, if

1. $S \subseteq B$ and
2. the inclusion map $\iota \colon S \to B$ preserves **and reflects** the constants, operations, and the predicates.

"Substructure" captures "image"

Definition. If $\mathbf{B}$ is a structure, then $\mathbf{S}$ is a **substructure** of $\mathbf{B}$, $\mathbf{S} \leq \mathbf{B}$, if

1. $S \subseteq B$ and
2. the inclusion map $\iota \colon S \to B$ preserves **and reflects** the constants, operations, and the predicates.

If the inclusion map preserves the constants and operations, then it will automatically reflect them.

“Substructure” captures “image”

Definition. If $\mathbf{B}$ is a structure, then $\mathbf{S}$ is a **substructure** of $\mathbf{B}$, $\mathbf{S} \leq \mathbf{B}$, if

1. $S \subseteq B$ and
2. the inclusion map $\iota \colon S \to B$ preserves **and reflects** the constants, operations, and the predicates.

If the inclusion map preserves the constants and operations, then it will automatically reflect them. But the inclusion could preserve the predicates without reflecting them.

"Substructure" captures "image"

Definition. If $\mathbf{B}$ is a structure, then $\mathbf{S}$ is a **substructure** of $\mathbf{B}$, $\mathbf{S} \leq \mathbf{B}$, if

1. $S \subseteq B$ and
2. the inclusion map $\iota\colon S \to B$ preserves **and reflects** the constants, operations, and the predicates.

If the inclusion map preserves the constants and operations, then it will automatically reflect them. But the inclusion could preserve the predicates without reflecting them. (Example?)

“Substructure” captures “image”

Definition. If $\mathbf{B}$ is a structure, then $\mathbf{S}$ is a **substructure** of $\mathbf{B}$, $\mathbf{S} \leq \mathbf{B}$, if

1. $S \subseteq B$ and
2. the inclusion map $\iota\colon S \to B$ preserves **and reflects** the constants, operations, and the predicates.

If the inclusion map preserves the constants and operations, then it will automatically reflect them. But the inclusion could preserve the predicates without reflecting them. (Example?)

Exercise.

“Substructure” captures “image”

Definition. If $\mathbf{B}$ is a structure, then $\mathbf{S}$ is a **substructure** of $\mathbf{B}$, $\mathbf{S} \leq \mathbf{B}$, if

1. $S \subseteq B$ and
2. the inclusion map $\iota\colon S \to B$ preserves **and reflects** the constants, operations, and the predicates.

If the inclusion map preserves the constants and operations, then it will automatically reflect them. But the inclusion could preserve the predicates without reflecting them. (Example?)

Exercise. Give an example of a poset $\langle P; \leq \rangle$,

"Substructure" captures "image"

Definition. If $\mathbf{B}$ is a structure, then $\mathbf{S}$ is a **substructure** of $\mathbf{B}$, $\mathbf{S} \leq \mathbf{B}$, if

1. $S \subseteq B$ and
2. the inclusion map $\iota \colon S \to B$ preserves **and reflects** the constants, operations, and the predicates.

If the inclusion map preserves the constants and operations, then it will automatically reflect them. But the inclusion could preserve the predicates without reflecting them. (Example?)

Exercise. Give an example of a poset $\langle P; \leq \rangle$, a subset $P' \subseteq P$,

“Substructure” captures “image”

Definition. If $\mathbf{B}$ is a structure, then $\mathbf{S}$ is a **substructure** of $\mathbf{B}$, $\mathbf{S} \leq \mathbf{B}$, if

1. $S \subseteq B$ and
2. the inclusion map $\iota\colon S \to B$ preserves **and reflects** the constants, operations, and the predicates.

If the inclusion map preserves the constants and operations, then it will automatically reflect them. But the inclusion could preserve the predicates without reflecting them. (Example?)

Exercise. Give an example of a poset $\langle P; \leq\rangle$, a subset $P' \subseteq P$, and a relation $\leq'$ $(\subseteq\leq)$ on P'

"Substructure" captures "image"

Definition. If $\mathbf{B}$ is a structure, then $\mathbf{S}$ is a **substructure** of $\mathbf{B}$, $\mathbf{S} \leq \mathbf{B}$, if

1. $S \subseteq B$ and
2. the inclusion map $\iota\colon S \to B$ preserves **and reflects** the constants, operations, and the predicates.

If the inclusion map preserves the constants and operations, then it will automatically reflect them. But the inclusion could preserve the predicates without reflecting them. (Example?)

Exercise. Give an example of a poset $\langle P; \leq \rangle$, a subset $P' \subseteq P$, and a relation $\leq'$ $(\subseteq \leq)$ on P' such that $(\forall x)(\forall y)((x \leq' y) \to (x \leq y))$,

“Substructure” captures “image”

Definition. If $\mathbf{B}$ is a structure, then $\mathbf{S}$ is a **substructure** of $\mathbf{B}$, $\mathbf{S} \leq \mathbf{B}$, if

1. $S \subseteq B$ and
2. the inclusion map $\iota \colon S \to B$ preserves **and reflects** the constants, operations, and the predicates.

If the inclusion map preserves the constants and operations, then it will automatically reflect them. But the inclusion could preserve the predicates without reflecting them. (Example?)

Exercise. Give an example of a poset $\langle P; \leq \rangle$, a subset $P' \subseteq P$, and a relation $\leq'$ $(\subseteq \leq)$ on P' such that $(\forall x)(\forall y)((x \leq' y) \to (x \leq y))$, where $\langle P'; \leq' \rangle$ is not a substructure of $\langle P; \leq \rangle$.

Embeddings

Embeddings

In the definition of ‘substructure’, the inclusion map played a special role.

Embeddings

In the definition of 'substructure', the inclusion map played a special role. If we replace the inclusion map with an arbitrary injective function, we obtain the definition of embedding.

Embeddings

In the definition of ‘substructure’, the inclusion map played a special role. If we replace the inclusion map with an arbitrary injective function, we obtain the definition of embedding.

Definition.

Embeddings

In the definition of ‘substructure’, the inclusion map played a special role. If we replace the inclusion map with an arbitrary injective function, we obtain the definition of embedding.

Definition. An **embedding**,

Embeddings

In the definition of ‘substructure’, the inclusion map played a special role. If we replace the inclusion map with an arbitrary injective function, we obtain the definition of embedding.

Definition. An **embedding**, $h\colon \mathbf{A} \to \mathbf{B}$,

Embeddings

In the definition of 'substructure', the inclusion map played a special role. If we replace the inclusion map with an arbitrary injective function, we obtain the definition of embedding.

Definition. An **embedding**, $h\colon \mathbf{A} \to \mathbf{B}$, is an injective function $h\colon A \to B$ that preserves and reflects the constants, operations, and the predicates.

Embeddings

In the definition of 'substructure', the inclusion map played a special role. If we replace the inclusion map with an arbitrary injective function, we obtain the definition of embedding.

Definition. An **embedding**, $h\colon \mathbf{A} \to \mathbf{B}$, is an injective function $h\colon A \to B$ that preserves and reflects the constants, operations, and the predicates.

Remark.

Embeddings

In the definition of ‘substructure’, the inclusion map played a special role. If we replace the inclusion map with an arbitrary injective function, we obtain the definition of embedding.

Definition. An **embedding**, $h\colon \mathbf{A} \to \mathbf{B}$, is an injective function $h\colon A \to B$ that preserves and reflects the constants, operations, and the predicates.

Remark. A bijective embedding is an isomorphism.

Embeddings

In the definition of ‘substructure’, the inclusion map played a special role. If we replace the inclusion map with an arbitrary injective function, we obtain the definition of embedding.

Definition. An **embedding**, $h\colon \mathbf{A} \to \mathbf{B}$, is an injective function $h\colon A \to B$ that preserves and reflects the constants, operations, and the predicates.

Remark. A bijective embedding is an isomorphism.

Test yourself!

Embeddings

In the definition of ‘substructure’, the inclusion map played a special role. If we replace the inclusion map with an arbitrary injective function, we obtain the definition of embedding.

Definition. An **embedding**, $h\colon \mathbf{A} \to \mathbf{B}$, is an injective function $h\colon A \to B$ that preserves and reflects the constants, operations, and the predicates.

Remark. A bijective embedding is an isomorphism.

Test yourself!

1. Give an example of an injective homomorphism of graphs that is not an embedding.

Embeddings

In the definition of ‘substructure’, the inclusion map played a special role. If we replace the inclusion map with an arbitrary injective function, we obtain the definition of embedding.

Definition. An **embedding**, $h\colon \mathbf{A} \to \mathbf{B}$, is an injective function $h\colon A \to B$ that preserves and reflects the constants, operations, and the predicates.

Remark. A bijective embedding is an isomorphism.

Test yourself!

1. Give an example of an injective homomorphism of graphs that is not an embedding.

Embeddings

In the definition of ‘substructure’, the inclusion map played a special role. If we replace the inclusion map with an arbitrary injective function, we obtain the definition of embedding.

Definition. An **embedding**, $h\colon \mathbf{A} \to \mathbf{B}$, is an injective function $h\colon A \to B$ that preserves and reflects the constants, operations, and the predicates.

Remark. A bijective embedding is an isomorphism.

Test yourself!

1. Give an example of an injective homomorphism of graphs that is not an embedding.
2. Give an example of an bijective homomorphism of posets that is not an isomorphism.

“Quotient” captures “coimage”, Part 1

“Quotient” captures “coimage”, Part 1

Definition.

“Quotient” captures “coimage”, Part 1

Definition. Let $\mathbf{A}$ be a structure and let θ be an equivalence relation on A.

“Quotient” captures “coimage”, Part 1

Definition. Let $\mathbf{A}$ be a structure and let θ be an equivalence relation on A. Call θ a **congruence** of $\mathbf{A}$ if for each operation symbol F we have

“Quotient” captures “coimage”, Part 1

Definition. Let $\mathbf{A}$ be a structure and let θ be an equivalence relation on A. Call θ a **congruence** of $\mathbf{A}$ if for each operation symbol F we have

$$a_1 \quad \equiv a_1' \pmod{\theta}$$

“Quotient” captures “coimage”, Part 1

Definition. Let $\mathbf{A}$ be a structure and let θ be an equivalence relation on A. Call θ a **congruence** of $\mathbf{A}$ if for each operation symbol F we have

$$\begin{aligned} a_1 &\equiv a_1' \pmod{\theta} \\ &\vdots \end{aligned}$$

“Quotient” captures “coimage”, Part 1

Definition. Let $\mathbf{A}$ be a structure and let θ be an equivalence relation on A. Call θ a **congruence** of $\mathbf{A}$ if for each operation symbol F we have

$$\begin{aligned} a_1 &\equiv a_1' \pmod{\theta} \\ &\vdots \\ a_n &\equiv a_n' \pmod{\theta} \end{aligned}$$

"Quotient" captures "coimage", Part 1

Definition. Let $\mathbf{A}$ be a structure and let θ be an equivalence relation on A. Call θ a **congruence** of $\mathbf{A}$ if for each operation symbol F we have

$$\begin{aligned} a_1 &\equiv a_1' \pmod{\theta} \\ &\vdots \\ a_n &\equiv a_n' \pmod{\theta} \end{aligned}$$

$$\Rightarrow \quad F^{\mathbf{A}}(a_1, \ldots, a_n) \equiv F^{\mathbf{A}}(a_1', \ldots, a_n') \pmod{\theta}$$

“Quotient” captures “coimage”, Part 2

“Quotient” captures “coimage”, Part 2

Definition.

“Quotient” captures “coimage”, Part 2

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A.

“Quotient” captures “coimage”, Part 2

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A. The **quotient** $\mathbf{A}/\theta$ has universe $A/\theta = \{a/\theta \mid a \in A\}$

“Quotient” captures “coimage”, Part 2

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A. The **quotient** $\mathbf{A}/\theta$ has universe $A/\theta = \{a/\theta \mid a \in A\} = \{[a]_\theta \mid a \in A\}$

“Quotient” captures “coimage”, Part 2

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A. The **quotient** $\mathbf{A}/\theta$ has universe $A/\theta = \{a/\theta \mid a \in A\} = \{[a]_\theta \mid a \in A\}$ and

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A. The **quotient** $\mathbf{A}/\theta$ has universe $A/\theta = \{a/\theta \mid a \in A\} = \{[a]_\theta \mid a \in A\}$ and

1. (Constants) $c^{\mathbf{A}/\theta} = c^{\mathbf{A}}/\theta$.

“Quotient” captures “coimage”, Part 2

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A. The **quotient** $\mathbf{A}/\theta$ has universe $A/\theta = \{a/\theta \mid a \in A\} = \{[a]_\theta \mid a \in A\}$ and

1. (Constants) $c^{\mathbf{A}/\theta} = c^{\mathbf{A}}/\theta$.

“Quotient” captures “coimage”, Part 2

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A. The **quotient** $\mathbf{A}/\theta$ has universe $A/\theta = \{a/\theta \mid a \in A\} = \{[a]_\theta \mid a \in A\}$ and

1. (Constants) $c^{\mathbf{A}/\theta} = c^{\mathbf{A}}/\theta$.
2. (Functions/Operations) $F^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = F^{\mathbf{A}}(a_1, \ldots, a_n)/\theta$.

"Quotient" captures "coimage", Part 2

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A. The **quotient** $\mathbf{A}/\theta$ has universe $A/\theta = \{a/\theta \mid a \in A\} = \{[a]_\theta \mid a \in A\}$ and

1. (Constants) $c^{\mathbf{A}/\theta} = c^{\mathbf{A}}/\theta$.
2. (Functions/Operations) $F^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = F^{\mathbf{A}}(a_1, \ldots, a_n)/\theta$.

“Quotient” captures “coimage”, Part 2

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A. The **quotient** $\mathbf{A}/\theta$ has universe $A/\theta = \{a/\theta \mid a \in A\} = \{[a]_\theta \mid a \in A\}$ and

1. (Constants) $c^{\mathbf{A}/\theta} = c^{\mathbf{A}}/\theta$.
2. (Functions/Operations) $F^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = F^{\mathbf{A}}(a_1, \ldots, a_n)/\theta$.
3. (Relations/Predicates) $R^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = \top$ iff $\exists a_i'$, $i = 1, \ldots, n$,

“Quotient” captures “coimage”, Part 2

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A. The **quotient** $\mathbf{A}/\theta$ has universe $A/\theta = \{a/\theta \mid a \in A\} = \{[a]_\theta \mid a \in A\}$ and

1. (Constants) $c^{\mathbf{A}/\theta} = c^{\mathbf{A}}/\theta$.
2. (Functions/Operations) $F^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = F^{\mathbf{A}}(a_1, \ldots, a_n)/\theta$.
3. (Relations/Predicates) $R^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = \top$ iff $\exists a_i'$, $i = 1, \ldots, n$,

“Quotient” captures “coimage”, Part 2

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A. The **quotient** $\mathbf{A}/\theta$ has universe $A/\theta = \{a/\theta \mid a \in A\} = \{[a]_\theta \mid a \in A\}$ and

1. (Constants) $c^{\mathbf{A}/\theta} = c^{\mathbf{A}}/\theta$.
2. (Functions/Operations) $F^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = F^{\mathbf{A}}(a_1, \ldots, a_n)/\theta$.
3. (Relations/Predicates) $R^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = \top$ iff $\exists a_i'$, $i = 1, \ldots, n$, such that $a_i \equiv a_i' \pmod{\theta}$ for all i

"Quotient" captures "coimage", Part 2

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A. The **quotient** $\mathbf{A}/\theta$ has universe $A/\theta = \{a/\theta \mid a \in A\} = \{[a]_\theta \mid a \in A\}$ and

1. (Constants) $c^{\mathbf{A}/\theta} = c^{\mathbf{A}}/\theta$.
2. (Functions/Operations) $F^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = F^{\mathbf{A}}(a_1, \ldots, a_n)/\theta$.
3. (Relations/Predicates) $R^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = \top$ iff $\exists a_i'$, $i = 1, \ldots, n$, such that $a_i \equiv a_i' \pmod{\theta}$ for all i and $R^{\mathbf{A}}(a_1', \ldots, a_n') = \top$.

“Quotient” captures “coimage”, Part 2

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A. The **quotient** $\mathbf{A}/\theta$ has universe $A/\theta = \{a/\theta \mid a \in A\} = \{[a]_\theta \mid a \in A\}$ and

1. (Constants) $c^{\mathbf{A}/\theta} = c^{\mathbf{A}}/\theta$.
2. (Functions/Operations) $F^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = F^{\mathbf{A}}(a_1, \ldots, a_n)/\theta$.
3. (Relations/Predicates) $R^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = \top$ iff $\exists a_i'$, $i = 1, \ldots, n$, such that $a_i \equiv a_i' \pmod{\theta}$ for all i and $R^{\mathbf{A}}(a_1', \ldots, a_n') = \top$.

Remark.

“Quotient” captures “coimage”, Part 2

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A. The **quotient** $\mathbf{A}/\theta$ has universe $A/\theta = \{a/\theta \mid a \in A\} = \{[a]_\theta \mid a \in A\}$ and

1. (Constants) $c^{\mathbf{A}/\theta} = c^{\mathbf{A}}/\theta$.
2. (Functions/Operations) $F^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = F^{\mathbf{A}}(a_1, \ldots, a_n)/\theta$.
3. (Relations/Predicates) $R^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = \top$ iff $\exists a_i'$, $i = 1, \ldots, n$, such that $a_i \equiv a_i' \pmod{\theta}$ for all i and $R^{\mathbf{A}}(a_1', \ldots, a_n') = \top$.

Remark. This definition of $\mathbf{A}/\theta$ makes the set A/θ the universe of an L-structure for which the natural map $\nu\colon A \to A/\theta\colon a \mapsto a/\theta$ is a homomorphism.

"Quotient" captures "coimage", Part 2

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A. The **quotient** $\mathbf{A}/\theta$ has universe $A/\theta = \{a/\theta \mid a \in A\} = \{[a]_\theta \mid a \in A\}$ and

1. (Constants) $c^{\mathbf{A}/\theta} = c^{\mathbf{A}}/\theta$.
2. (Functions/Operations) $F^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = F^{\mathbf{A}}(a_1, \ldots, a_n)/\theta$.
3. (Relations/Predicates) $R^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = \top$ iff $\exists a_i'$, $i = 1, \ldots, n$, such that $a_i \equiv a_i' \pmod{\theta}$ for all i and $R^{\mathbf{A}}(a_1', \ldots, a_n') = \top$.

Remark. This definition of $\mathbf{A}/\theta$ makes the set A/θ the universe of an L-structure for which the natural map $\nu\colon A \to A/\theta\colon a \mapsto a/\theta$ is a homomorphism. Moreover, the quotient uses the weakest interpretation of the predicates that makes the natural map a homomorphism.

“Quotient” captures “coimage”, Part 2

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A. The **quotient** $\mathbf{A}/\theta$ has universe $A/\theta = \{a/\theta \mid a \in A\} = \{[a]_\theta \mid a \in A\}$ and

1. (Constants) $c^{\mathbf{A}/\theta} = c^{\mathbf{A}}/\theta$.
2. (Functions/Operations) $F^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = F^{\mathbf{A}}(a_1, \ldots, a_n)/\theta$.
3. (Relations/Predicates) $R^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = \top$ iff $\exists a_i'$, $i = 1, \ldots, n$, such that $a_i \equiv a_i' \pmod{\theta}$ for all i and $R^{\mathbf{A}}(a_1', \ldots, a_n') = \top$.

Remark. This definition of $\mathbf{A}/\theta$ makes the set A/θ the universe of an L-structure for which the natural map $\nu\colon A \to A/\theta\colon a \mapsto a/\theta$ is a homomorphism. Moreover, the quotient uses the weakest interpretation of the predicates that makes the natural map a homomorphism.

Exercise.

“Quotient” captures “coimage”, Part 2

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A. The **quotient** $\mathbf{A}/\theta$ has universe $A/\theta = \{a/\theta \mid a \in A\} = \{[a]_\theta \mid a \in A\}$ and

1. (Constants) $c^{\mathbf{A}/\theta} = c^{\mathbf{A}}/\theta$.
2. (Functions/Operations) $F^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = F^{\mathbf{A}}(a_1, \ldots, a_n)/\theta$.
3. (Relations/Predicates) $R^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = \top$ iff $\exists a_i'$, $i = 1, \ldots, n$, such that $a_i \equiv a_i' \pmod{\theta}$ for all i and $R^{\mathbf{A}}(a_1', \ldots, a_n') = \top$.

Remark. This definition of $\mathbf{A}/\theta$ makes the set A/θ the universe of an L-structure for which the natural map $\nu\colon A \to A/\theta\colon a \mapsto a/\theta$ is a homomorphism. Moreover, the quotient uses the weakest interpretation of the predicates that makes the natural map a homomorphism.

Exercise. Find all the quotients (up to isomorphism)

"Quotient" captures "coimage", Part 2

Definition. Let $\mathbf{A}$ be an L-structure and let θ be a congruence on A. The **quotient** $\mathbf{A}/\theta$ has universe $A/\theta = \{a/\theta \mid a \in A\} = \{[a]_\theta \mid a \in A\}$ and

1. (Constants) $c^{\mathbf{A}/\theta} = c^{\mathbf{A}}/\theta$.
2. (Functions/Operations) $F^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = F^{\mathbf{A}}(a_1, \ldots, a_n)/\theta$.
3. (Relations/Predicates) $R^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = \top$ iff $\exists a_i'$, $i = 1, \ldots, n$, such that $a_i \equiv a_i' \pmod{\theta}$ for all i and $R^{\mathbf{A}}(a_1', \ldots, a_n') = \top$.

Remark. This definition of $\mathbf{A}/\theta$ makes the set A/θ the universe of an L-structure for which the natural map $\nu\colon A \to A/\theta\colon a \mapsto a/\theta$ is a homomorphism. Moreover, the quotient uses the weakest interpretation of the predicates that makes the natural map a homomorphism.

Exercise. Find all the quotients (up to isomorphism) of the symmetric graph $\langle V; E(x,y)\rangle$ that is a 4-element path.

Products

Products

Definition.

Products

Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures.

Products

Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures. The **(Cartesian) product** of this indexed set is the L-structure $\mathbf{P}$

Products

Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures. The **(Cartesian) product** of this indexed set is the L-structure $\mathbf{P}$ whose universe is $P = \prod_{i \in I} A_i$,

Products

Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures. The **(Cartesian) product** of this indexed set is the L-structure $\mathbf{P}$ whose universe is $P = \prod_{i \in I} A_i$, the Cartesian product of the universes of members of $\mathcal{K}$.

Products

Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures. The **(Cartesian) product** of this indexed set is the L-structure $\mathbf{P}$ whose universe is $P = \prod_{i \in I} A_i$, the Cartesian product of the universes of members of $\mathcal{K}$. We choose the interpretations of the constants, operations, and predicates in the weakest possible way to ensure that all of the coordinate projection functions $\pi_i \colon \prod_{i \in I} A_i \to A_i$ are homomorphisms $\pi_i \colon \mathbf{P} \to \mathbf{A}_i$.

Products

Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures. The **(Cartesian) product** of this indexed set is the L-structure $\mathbf{P}$ whose universe is $P = \prod_{i\in I} A_i$, the Cartesian product of the universes of members of $\mathcal{K}$. We choose the interpretations of the constants, operations, and predicates in the weakest possible way to ensure that all of the coordinate projection functions $\pi_i \colon \prod_{i\in I} A_i \to A_i$ are homomorphisms $\pi_i \colon \mathbf{P} \to \mathbf{A}_i$. Specifically,

Products

Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures. The **(Cartesian) product** of this indexed set is the L-structure $\mathbf{P}$ whose universe is $P = \prod_{i \in I} A_i$, the Cartesian product of the universes of members of $\mathcal{K}$. We choose the interpretations of the constants, operations, and predicates in the weakest possible way to ensure that all of the coordinate projection functions $\pi_i \colon \prod_{i \in I} A_i \to A_i$ are homomorphisms $\pi_i \colon \mathbf{P} \to \mathbf{A}_i$. Specifically,

1. (Constants) $c^{\prod_{i \in I} \mathbf{A}_i} = (c^{\mathbf{A}_i})_{i \in I}$.

Products

Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures. The **(Cartesian) product** of this indexed set is the L-structure $\mathbf{P}$ whose universe is $P = \prod_{i\in I} A_i$, the Cartesian product of the universes of members of $\mathcal{K}$. We choose the interpretations of the constants, operations, and predicates in the weakest possible way to ensure that all of the coordinate projection functions $\pi_i \colon \prod_{i\in I} A_i \to A_i$ are homomorphisms $\pi_i \colon \mathbf{P} \to \mathbf{A}_i$. Specifically,

1. (Constants) $c^{\prod_{i\in I} \mathbf{A}_i} = (c^{\mathbf{A}_i})_{i\in I}$.

Products

Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures. The **(Cartesian) product** of this indexed set is the L-structure $\mathbf{P}$ whose universe is $P = \prod_{i\in I} A_i$, the Cartesian product of the universes of members of $\mathcal{K}$. We choose the interpretations of the constants, operations, and predicates in the weakest possible way to ensure that all of the coordinate projection functions $\pi_i \colon \prod_{i\in I} A_i \to A_i$ are homomorphisms $\pi_i \colon \mathbf{P} \to \mathbf{A}_i$. Specifically,

1. (Constants) $c^{\prod_{i\in I}\mathbf{A}_i} = (c^{\mathbf{A}_i})_{i\in I}$.
2. (Operations) $F^{\prod_{i\in I}\mathbf{A}_i}((a_{1i})_{i\in I}, \ldots, (a_{ni})_{i\in I}) = (F^{\mathbf{A}_i}(a_{1i}, \ldots, a_{ni}))_{i\in I}$.

Products

Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures. The **(Cartesian) product** of this indexed set is the L-structure $\mathbf{P}$ whose universe is $P = \prod_{i \in I} A_i$, the Cartesian product of the universes of members of $\mathcal{K}$. We choose the interpretations of the constants, operations, and predicates in the weakest possible way to ensure that all of the coordinate projection functions $\pi_i \colon \prod_{i \in I} A_i \to A_i$ are homomorphisms $\pi_i \colon \mathbf{P} \to \mathbf{A}_i$. Specifically,

1. (Constants) $c^{\prod_{i \in I} \mathbf{A}_i} = (c^{\mathbf{A}_i})_{i \in I}$.
2. (Operations) $F^{\prod_{i \in I} \mathbf{A}_i}((a_{1i})_{i \in I}, \ldots, (a_{ni})_{i \in I}) = (F^{\mathbf{A}_i}(a_{1i}, \ldots, a_{ni}))_{i \in I}$.

Products

Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures. The **(Cartesian) product** of this indexed set is the L-structure $\mathbf{P}$ whose universe is $P = \prod_{i \in I} A_i$, the Cartesian product of the universes of members of $\mathcal{K}$. We choose the interpretations of the constants, operations, and predicates in the weakest possible way to ensure that all of the coordinate projection functions $\pi_i \colon \prod_{i \in I} A_i \to A_i$ are homomorphisms $\pi_i \colon \mathbf{P} \to \mathbf{A}_i$. Specifically,

1. (Constants) $c^{\prod_{i \in I} \mathbf{A}_i} = (c^{\mathbf{A}_i})_{i \in I}$.
2. (Operations)
   $F^{\prod_{i \in I} \mathbf{A}_i}((a_{1i})_{i \in I}, \ldots, (a_{ni})_{i \in I}) = (F^{\mathbf{A}_i}(a_{1i}, \ldots, a_{ni}))_{i \in I}$.
3. (Predicates) $R^{\prod_{i \in I} \mathbf{A}_i}((a_{1i})_{i \in I}, \ldots, (a_{ni})_{i \in I}) = \top$ iff
   $R^{\mathbf{A}_i}(a_{1i}, \ldots, a_{ni}) = \top$ for all i.

Products

Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures. The **(Cartesian) product** of this indexed set is the L-structure $\mathbf{P}$ whose universe is $P = \prod_{i \in I} A_i$, the Cartesian product of the universes of members of $\mathcal{K}$. We choose the interpretations of the constants, operations, and predicates in the weakest possible way to ensure that all of the coordinate projection functions $\pi_i \colon \prod_{i \in I} A_i \to A_i$ are homomorphisms $\pi_i \colon \mathbf{P} \to \mathbf{A}_i$. Specifically,

1. (Constants) $c^{\prod_{i \in I} \mathbf{A}_i} = (c^{\mathbf{A}_i})_{i \in I}$.
2. (Operations) $F^{\prod_{i \in I} \mathbf{A}_i}((a_{1i})_{i \in I}, \ldots, (a_{ni})_{i \in I}) = (F^{\mathbf{A}_i}(a_{1i}, \ldots, a_{ni}))_{i \in I}$.
3. (Predicates) $R^{\prod_{i \in I} \mathbf{A}_i}((a_{1i})_{i \in I}, \ldots, (a_{ni})_{i \in I}) = \top$ iff $R^{\mathbf{A}_i}(a_{1i}, \ldots, a_{ni}) = \top$ for all i.

Products

Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures. The **(Cartesian) product** of this indexed set is the L-structure $\mathbf{P}$ whose universe is $P = \prod_{i \in I} A_i$, the Cartesian product of the universes of members of $\mathcal{K}$. We choose the interpretations of the constants, operations, and predicates in the weakest possible way to ensure that all of the coordinate projection functions $\pi_i \colon \prod_{i \in I} A_i \to A_i$ are homomorphisms $\pi_i \colon \mathbf{P} \to \mathbf{A}_i$. Specifically,

1. (Constants) $c^{\prod_{i \in I} \mathbf{A}_i} = (c^{\mathbf{A}_i})_{i \in I}$.
2. (Operations)
   $F^{\prod_{i \in I} \mathbf{A}_i}((a_{1i})_{i \in I}, \ldots, (a_{ni})_{i \in I}) = (F^{\mathbf{A}_i}(a_{1i}, \ldots, a_{ni}))_{i \in I}$.
3. (Predicates) $R^{\prod_{i \in I} \mathbf{A}_i}((a_{1i})_{i \in I}, \ldots, (a_{ni})_{i \in I}) = \top$ iff
   $R^{\mathbf{A}_i}(a_{1i}, \ldots, a_{ni}) = \top$ for all i.

Exercise.

Products

Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures. The **(Cartesian) product** of this indexed set is the L-structure $\mathbf{P}$ whose universe is $P = \prod_{i\in I} A_i$, the Cartesian product of the universes of members of $\mathcal{K}$. We choose the interpretations of the constants, operations, and predicates in the weakest possible way to ensure that all of the coordinate projection functions $\pi_i \colon \prod_{i\in I} A_i \to A_i$ are homomorphisms $\pi_i \colon \mathbf{P} \to \mathbf{A}_i$. Specifically,

1. (Constants) $c^{\prod_{i\in I}\mathbf{A}_i} = (c^{\mathbf{A}_i})_{i\in I}$.
2. (Operations) $F^{\prod_{i\in I}\mathbf{A}_i}((a_{1i})_{i\in I}, \ldots, (a_{ni})_{i\in I}) = (F^{\mathbf{A}_i}(a_{1i}, \ldots, a_{ni}))_{i\in I}$.
3. (Predicates) $R^{\prod_{i\in I}\mathbf{A}_i}((a_{1i})_{i\in I}, \ldots, (a_{ni})_{i\in I}) = \top$ iff $R^{\mathbf{A}_i}(a_{1i}, \ldots, a_{ni}) = \top$ for all i.

Exercise. Describe the product of two symmetric, loopless graphs,

Products

Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures. The **(Cartesian) product** of this indexed set is the L-structure $\mathbf{P}$ whose universe is $P = \prod_{i\in I} A_i$, the Cartesian product of the universes of members of $\mathcal{K}$. We choose the interpretations of the constants, operations, and predicates in the weakest possible way to ensure that all of the coordinate projection functions $\pi_i\colon \prod_{i\in I} A_i \to A_i$ are homomorphisms $\pi_i\colon \mathbf{P} \to \mathbf{A}_i$. Specifically,

1. (Constants) $c^{\prod_{i\in I}\mathbf{A}_i} = (c^{\mathbf{A}_i})_{i\in I}$.
2. (Operations) $F^{\prod_{i\in I}\mathbf{A}_i}((a_{1i})_{i\in I}, \ldots, (a_{ni})_{i\in I}) = (F^{\mathbf{A}_i}(a_{1i}, \ldots, a_{ni}))_{i\in I}$.
3. (Predicates) $R^{\prod_{i\in I}\mathbf{A}_i}((a_{1i})_{i\in I}, \ldots, (a_{ni})_{i\in I}) = \top$ iff $R^{\mathbf{A}_i}(a_{1i}, \ldots, a_{ni}) = \top$ for all i.

Exercise. Describe the product of two symmetric, loopless graphs, which are both 3-vertex paths.

Products

Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures. The **(Cartesian) product** of this indexed set is the L-structure $\mathbf{P}$ whose universe is $P = \prod_{i \in I} A_i$, the Cartesian product of the universes of members of $\mathcal{K}$. We choose the interpretations of the constants, operations, and predicates in the weakest possible way to ensure that all of the coordinate projection functions $\pi_i \colon \prod_{i \in I} A_i \to A_i$ are homomorphisms $\pi_i \colon \mathbf{P} \to \mathbf{A}_i$. Specifically,

1. (Constants) $c^{\prod_{i \in I} \mathbf{A}_i} = (c^{\mathbf{A}_i})_{i \in I}$.
2. (Operations) $F^{\prod_{i \in I} \mathbf{A}_i}((a_{1i})_{i \in I}, \ldots, (a_{ni})_{i \in I}) = (F^{\mathbf{A}_i}(a_{1i}, \ldots, a_{ni}))_{i \in I}$.
3. (Predicates) $R^{\prod_{i \in I} \mathbf{A}_i}((a_{1i})_{i \in I}, \ldots, (a_{ni})_{i \in I}) = \top$ iff $R^{\mathbf{A}_i}(a_{1i}, \ldots, a_{ni}) = \top$ for all i.

Exercise. Describe the product of two symmetric, loopless graphs, which are both 3-vertex paths. Try this exercise again when the graphs have loops on every vertex.

Reduced products and ultraproducts, Part 1

Reduced products and ultraproducts, Part 1

Definitions.

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

Reduced products and ultraproducts, Part 1

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward)

Reduced products and ultraproducts, Part 1

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward)

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$

Reduced products and ultraproducts, Part 1

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection)

Reduced products and ultraproducts, Part 1

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection)

Reduced products and ultraproducts, Part 1

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Related terms.

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Related terms.

1. A filter is **proper** if $\mathcal{F} \neq \mathcal{P}(I)$

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Related terms.

1. A filter is **proper** if $\mathcal{F} \neq \mathcal{P}(I)$

Reduced products and ultraproducts, Part 1

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Related terms.

1. A filter is **proper** if $\mathcal{F} \neq \mathcal{P}(I)$ (equivalently $\emptyset \notin \mathcal{F}$).

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Related terms.

1. A filter is **proper** if $\mathcal{F} \neq \mathcal{P}(I)$ (equivalently $\emptyset \notin \mathcal{F}$). Otherwise the filter is **improper**.

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Related terms.

1. A filter is **proper** if $\mathcal{F} \neq \mathcal{P}(I)$ (equivalently $\emptyset \notin \mathcal{F}$). Otherwise the filter is **improper**.
2. A filter is **principal** if there is some subset $I_0 \subseteq I$ such that $\mathcal{F}$ is the set of all subsets of I containing I_0.

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Related terms.

1. A filter is **proper** if $\mathcal{F} \neq \mathcal{P}(I)$ (equivalently $\emptyset \notin \mathcal{F}$). Otherwise the filter is **improper**.
2. A filter is **principal** if there is some subset $I_0 \subseteq I$ such that $\mathcal{F}$ is the set of all subsets of I containing I_0.

Reduced products and ultraproducts, Part 1

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Related terms.

1. A filter is **proper** if $\mathcal{F} \neq \mathcal{P}(I)$ (equivalently $\emptyset \notin \mathcal{F}$). Otherwise the filter is **improper**.
2. A filter is **principal** if there is some subset $I_0 \subseteq I$ such that $\mathcal{F}$ is the set of all subsets of I containing I_0. $\mathcal{F} = (I_0)$.

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Related terms.

1. A filter is **proper** if $\mathcal{F} \neq \mathcal{P}(I)$ (equivalently $\emptyset \notin \mathcal{F}$). Otherwise the filter is **improper**.
2. A filter is **principal** if there is some subset $I_0 \subseteq I$ such that $\mathcal{F}$ is the set of all subsets of I containing I_0. $\mathcal{F} = (I_0)$. Otherwise it is **nonprincipal**.

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Related terms.

1. A filter is **proper** if $\mathcal{F} \neq \mathcal{P}(I)$ (equivalently $\emptyset \notin \mathcal{F}$). Otherwise the filter is **improper**.
2. A filter is **principal** if there is some subset $I_0 \subseteq I$ such that $\mathcal{F}$ is the set of all subsets of I containing I_0. $\mathcal{F} = (I_0)$. Otherwise it is **nonprincipal**.
3. A filter is **an ultrafilter** if it is proper

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Related terms.

1. A filter is **proper** if $\mathcal{F} \neq \mathcal{P}(I)$ (equivalently $\emptyset \notin \mathcal{F}$). Otherwise the filter is **improper**.
2. A filter is **principal** if there is some subset $I_0 \subseteq I$ such that $\mathcal{F}$ is the set of all subsets of I containing I_0. $\mathcal{F} = (I_0)$. Otherwise it is **nonprincipal**.
3. A filter is **an ultrafilter** if it is proper

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Related terms.

1. A filter is **proper** if $\mathcal{F} \neq \mathcal{P}(I)$ (equivalently $\emptyset \notin \mathcal{F}$). Otherwise the filter is **improper**.
2. A filter is **principal** if there is some subset $I_0 \subseteq I$ such that $\mathcal{F}$ is the set of all subsets of I containing I_0. $\mathcal{F} = (I_0)$. Otherwise it is **nonprincipal**.
3. A filter is **an ultrafilter** if it is proper and for every subset $U \subseteq I$

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Related terms.

1. A filter is **proper** if $\mathcal{F} \neq \mathcal{P}(I)$ (equivalently $\emptyset \notin \mathcal{F}$). Otherwise the filter is **improper**.
2. A filter is **principal** if there is some subset $I_0 \subseteq I$ such that $\mathcal{F}$ is the set of all subsets of I containing I_0. $\mathcal{F} = (I_0)$. Otherwise it is **nonprincipal**.
3. A filter is **an ultrafilter** if it is proper and for every subset $U \subseteq I$ either $U \in \mathcal{F}$

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Related terms.

1. A filter is **proper** if $\mathcal{F} \neq \mathcal{P}(I)$ (equivalently $\emptyset \notin \mathcal{F}$). Otherwise the filter is **improper**.
2. A filter is **principal** if there is some subset $I_0 \subseteq I$ such that $\mathcal{F}$ is the set of all subsets of I containing I_0. $\mathcal{F} = (I_0)$. Otherwise it is **nonprincipal**.
3. A filter is **an ultrafilter** if it is proper and for every subset $U \subseteq I$ either $U \in \mathcal{F}$ or $I \setminus U \in \mathcal{F}$.

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Related terms.

1. A filter is **proper** if $\mathcal{F} \neq \mathcal{P}(I)$ (equivalently $\emptyset \notin \mathcal{F}$). Otherwise the filter is **improper**.
2. A filter is **principal** if there is some subset $I_0 \subseteq I$ such that $\mathcal{F}$ is the set of all subsets of I containing I_0. $\mathcal{F} = (I_0)$. Otherwise it is **nonprincipal**.
3. A filter is **an ultrafilter** if it is proper and for every subset $U \subseteq I$ either $U \in \mathcal{F}$ or $I \setminus U \in \mathcal{F}$. (Check:

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Related terms.

1. A filter is **proper** if $\mathcal{F} \neq \mathcal{P}(I)$ (equivalently $\emptyset \notin \mathcal{F}$). Otherwise the filter is **improper**.
2. A filter is **principal** if there is some subset $I_0 \subseteq I$ such that $\mathcal{F}$ is the set of all subsets of I containing I_0. $\mathcal{F} = (I_0)$. Otherwise it is **nonprincipal**.
3. A filter is **an ultrafilter** if it is proper and for every subset $U \subseteq I$ either $U \in \mathcal{F}$ or $I \setminus U \in \mathcal{F}$. (Check: An ultrafilter is principal if and only if $\mathcal{F} = (I_0)$ where $I_0 = \{i_0\}$ is a singleton.)

Definitions. Let I be a set. A **filter** on I is a nonempty subset $\mathcal{F} \subseteq \mathcal{P}(I)$ that is

1. (Closed upward) $U \in \mathcal{F}$ and $U \subseteq V$ jointly imply that $V \in \mathcal{F}$.
2. (Closed under finite intersection) If $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$.

Related terms.

1. A filter is **proper** if $\mathcal{F} \neq \mathcal{P}(I)$ (equivalently $\emptyset \notin \mathcal{F}$). Otherwise the filter is **improper**.
2. A filter is **principal** if there is some subset $I_0 \subseteq I$ such that $\mathcal{F}$ is the set of all subsets of I containing I_0. $\mathcal{F} = (I_0)$. Otherwise it is **nonprincipal**.
3. A filter is **an ultrafilter** if it is proper and for every subset $U \subseteq I$ either $U \in \mathcal{F}$ or $I \setminus U \in \mathcal{F}$. (Check: An ultrafilter is principal if and only if $\mathcal{F} = (I_0)$ where $I_0 = \{i_0\}$ is a singleton.)

Reduced products and ultraproducts, Part 2

(Bad!) Definition.

(Bad!) Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures

(Bad!) Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures and let $\mathcal{F}$ be a filter on I.

Reduced products and ultraproducts, Part 2

(Bad!) Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures and let $\mathcal{F}$ be a filter on I. The binary relation $\theta_{\mathcal{F}}$ on $\prod_{i \in I} A_i$ defined by

$$(\mathbf{a}, \mathbf{a}') \in \theta_{\mathcal{F}} \quad \text{iff} \quad [\![\mathbf{a} = \mathbf{a}']\!] \in \mathcal{F}$$

Reduced products and ultraproducts, Part 2

(Bad!) Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures and let $\mathcal{F}$ be a filter on I. The binary relation $\theta_{\mathcal{F}}$ on $\prod_{i \in I} A_i$ defined by

$$(\mathbf{a}, \mathbf{a}') \in \theta_{\mathcal{F}} \quad \text{iff} \quad [\![\mathbf{a} = \mathbf{a}']\!] \in \mathcal{F}$$

is a congruence on $\prod_{i \in I} \mathbf{A}_i$

(Bad!) Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures and let $\mathcal{F}$ be a filter on I. The binary relation $\theta_{\mathcal{F}}$ on $\prod_{i \in I} A_i$ defined by

$$(\mathbf{a}, \mathbf{a}') \in \theta_{\mathcal{F}} \quad \text{iff} \quad [\![\mathbf{a} = \mathbf{a}']\!] \in \mathcal{F}$$

is a congruence on $\prod_{i \in I} \mathbf{A}_i$ called the **filter congruence** associated to $\mathcal{F}$.

(Bad!) Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures and let $\mathcal{F}$ be a filter on I. The binary relation $\theta_{\mathcal{F}}$ on $\prod_{i\in I} A_i$ defined by

$$(\mathbf{a}, \mathbf{a}') \in \theta_{\mathcal{F}} \quad \text{iff} \quad [\![\mathbf{a} = \mathbf{a}']\!] \in \mathcal{F}$$

is a congruence on $\prod_{i\in I} \mathbf{A}_i$ called the **filter congruence** associated to $\mathcal{F}$.
(Check that it is a congruence!)

Reduced products and ultraproducts, Part 2

(Bad!) Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures and let $\mathcal{F}$ be a filter on I. The binary relation $\theta_{\mathcal{F}}$ on $\prod_{i \in I} A_i$ defined by

$$(\mathbf{a}, \mathbf{a}') \in \theta_{\mathcal{F}} \quad \text{iff} \quad [\![\mathbf{a} = \mathbf{a}']\!] \in \mathcal{F}$$

is a congruence on $\prod_{i \in I} \mathbf{A}_i$ called the **filter congruence** associated to $\mathcal{F}$. (Check that it is a congruence!) The quotient $(\prod_{i \in I} \mathbf{A}_i)/\theta_{\mathcal{F}}$ is a **reduced product** of the algebras in $\mathcal{K}$.

Reduced products and ultraproducts, Part 2

(Bad!) Definition. Let $\mathcal{K} = \{\mathbf{A}_i \mid i \in I\}$ be an indexed set of L-structures and let $\mathcal{F}$ be a filter on I. The binary relation $\theta_{\mathcal{F}}$ on $\prod_{i \in I} A_i$ defined by

$$(\mathbf{a}, \mathbf{a}') \in \theta_{\mathcal{F}} \quad \text{iff} \quad [\![\mathbf{a} = \mathbf{a}']\!] \in \mathcal{F}$$

is a congruence on $\prod_{i \in I} \mathbf{A}_i$ called the **filter congruence** associated to $\mathcal{F}$. (Check that it is a congruence!) The quotient $(\prod_{i \in I} \mathbf{A}_i)/\theta_{\mathcal{F}}$ is a **reduced product** of the algebras in $\mathcal{K}$. If $\mathcal{F}$ is an ultrafilter, then the reduced product $(\prod_{i \in I} \mathbf{A}_i)/\theta_{\mathcal{F}}$ is called an **ultraproduct** of the algebras in $\mathcal{K}$.