

The Completeness Theorem

$$\Sigma \models \sigma \text{ iff } \Sigma \vdash \sigma$$



Nature decides truth

The relation \models defines a Galois connection between L -structures and L -sentences.

We write $\Sigma \models \sigma$ to indicate that σ lies in the Galois closure of Σ .
(i.e. $\sigma \in \Sigma^{\perp\perp}$).

How can we characterize the Galois closure of Σ “internally”? (meaning: how can you determine whether $\sigma \in \Sigma^{\perp\perp}$ without referring to structures?)

Humans decide provability

We create a machine called “proof”, where σ is provable from Σ ($\Sigma \vdash \sigma$) iff σ is a semantic consequence of Σ ($\Sigma \models \sigma$).

If our only goal is to characterize Galois closure internally, then we only demand that our proof calculus be

- Sound ($\Sigma \vdash \sigma$ implies $\Sigma \models \sigma$), and
- Complete ($\Sigma \models \sigma$ implies $\Sigma \vdash \sigma$).

But, since we are humans, we shall also demand that

- proofs should be recognizable as proofs.

Df. $\Sigma \vdash \sigma$ means there is a finite sequence of formulas

$$\alpha_1, \alpha_2, \dots, \alpha_n = \sigma$$

where each α_i is an **axiom**, a member of Σ , or is derivable from earlier terms in the sequence using a **rule of inference**.

What is needed?

We should choose **axioms** so that they are recognizable instances of $\models \alpha$.

We should choose rules of inference, typically written $\frac{\alpha_1, \dots, \alpha_m}{\beta}$, so that they are recognizable instances of $\{\alpha_1, \dots, \alpha_m\} \models \beta$.

“Should” means: if we do this, then soundness will hold.

Axioms.

- 1 All tautologies.
- 2 $=$ is an equivalence relation on terms.
- 3 Can substitute equals for equals without changing meaning.
- 4 $(\forall x_i(\alpha \rightarrow \beta)) \rightarrow (\forall x_i \alpha \rightarrow \forall x_i \beta)$
- 5 $(\alpha \rightarrow \forall x_i \alpha)$ if x_i does not appear in formula α .
- 6 $(\exists x_i(x_i = t))$ if x_i does not occur in term t .

Rules.

- 1 (Modus Ponens) $\frac{\alpha, \alpha \rightarrow \beta}{\beta}$
- 2 (Generalization) $\frac{\varphi}{(\forall x_i) \varphi}$

Stage 1: the Deduction Theorem

Observe that $\Sigma \models \sigma$ iff $\Sigma \cup \{\neg\sigma\} \models \perp$. (Note: $\forall A (A \not\models \perp)$. I.e., \perp is not satisfiable.)

Therefore we want $\Sigma \vdash \sigma$ iff $\Sigma \cup \{\neg\sigma\} \vdash \perp$.

“More generally”, $\Sigma \cup \{\alpha\} \models \beta$ iff $\Sigma \models (\alpha \rightarrow \beta)$.

So we want $\Sigma \cup \{\alpha\} \vdash \beta$ iff $\Sigma \vdash (\alpha \rightarrow \beta)$.

Reverse direction is direct and easy.

Forward direction is proved by induction on the length of a proof of $\Sigma \cup \{\alpha\} \vdash \beta$.

It is also easy.

[Idea: Replace every α_i in a $(\Sigma \cup \{\alpha\})$ -proof of β with $\alpha \rightarrow \alpha_i$ to obtain a Σ -proof of $(\alpha \rightarrow \beta)$.]

The second part is called:

Deduction Thm. If $\Sigma \cup \{\alpha\} \models \beta$, then $\Sigma \models (\alpha \rightarrow \beta)$.

Application

Our goal is to prove that $\Sigma \models \sigma$ implies $\Sigma \vdash \sigma$.

Equivalently, if $\Gamma := \Sigma \cup \{\neg\sigma\}$ is not **satisfiable** ($\Gamma \models \perp$), then it is not **consistent** ($\Gamma \vdash \perp$).

Contrapositively, if Γ is consistent, then it is satisfiable (i.e. has a model).
(This reformulation is worth remembering!)

Strategy to achieve our goal:

- 1 Show that a consistent theory Γ can be enlarged to a “Henkin theory”.
- 2 Show that a Henkin theory has a model.
- 3 Show that a model of an enlargement of Γ is also a model of Γ . (Duh!)

Henkin theory

Df. A theory Γ is a Henkin theory if it is

- ① **consistent**,
- ② **complete**, and
- ③ **has witnesses**.

Meanings:

- ① A theory is **consistent** if you can't prove falsity from it.
- ② A consistent theory Γ is **complete** if it decides every sentence:
For every σ , either $\sigma \in \Gamma$ or $(\neg\sigma) \in \Gamma$.
- ③ A theory Γ **has witnesses** if whenever $\varphi(x)$ is a formula with one free variable, then $((\exists x)\varphi(x) \rightarrow \varphi(c)) \in \Gamma$ for some constant c .

Henkin's key insight is that if \mathbb{A} is a structure, then the theory of its “expansion by constants”, $\Gamma = \text{Th}(\mathbb{A}_A)$, is a Henkin theory. Conversely every Henkin theory arises in this way. Moreover, $\text{Th}(\mathbb{A}_A)$ explains clearly how to construct its canonical model, \mathbb{A}_A .

The enlargement steps

Lindenbaum's Theorem. Every consistent L -theory can be enlarged to a complete L -theory.

[Idea: if $\Gamma \not\vdash \sigma$, then $\Gamma \cup \{\neg\sigma\} \not\vdash \perp$, so $\Gamma \cup \{\neg\sigma\}$ is a consistent enlargement of Γ . Keep doing this until you arrive at a complete theory.]

Henkin's Theorem. Every consistent theory can be enlarged to a consistent theory with witnesses, provided we allow ourselves to enlarge the language to include more constant symbols.

[Idea: suppose $\Gamma \cup \{(\exists x)\varphi(x) \rightarrow \varphi(c)\} \vdash \perp$ where $c \notin L$. Then $\Gamma \vdash \neg((\exists x)\varphi(x) \rightarrow \varphi(c))$, or $\Gamma \vdash (\exists x)\varphi(x) \wedge \neg\varphi(c)$. Need quantifier axioms and rules which permit this deduction:

$$(\exists x)\varphi(x) \wedge \neg\varphi(c), (\forall x)((\exists x)\varphi(x) \wedge \neg\varphi(x)), (\exists x)\varphi(x) \wedge \neg(\exists x)\varphi(x), \perp.$$

Thus $\Gamma \vdash \perp$. Now repeat the idea of Lindenbaum's Theorem with σ equal to $\neg((\exists x)\varphi(x) \rightarrow \varphi(c))$.]

Finally: Henkin theories have an obvious model.

Let H be a Henkin L -theory. (= consistent, complete, with witnesses.)

Let C be the set of constants in L . It will be the domain of an L -structure.

If $c \in L$, then define $c^{\mathbb{C}} = c \in C$.

If $R(x_1, \dots, x_n)$ is a predicate symbol, declare that $R^{\mathbb{C}}(c_1, \dots, c_n)$ is true if $R(c_1, \dots, c_n) \in H$.

If $F(x_1, \dots, x_n)$ is a function symbol, declare that $F^{\mathbb{C}}(c_1, \dots, c_n) = d$ is true if $(F(c_1, \dots, c_n) = d) \in H$.

Define an equivalence relation θ on C by $c \equiv d \pmod{\theta}$ if $(c = d) \in H$.

It will be the case that $\mathbb{C}/\theta \models H$. In fact, $H = \text{Th}(\mathbb{C}/\theta)$. \mathbb{C}/θ is called the Henkin model of H .