The Completeness Theorem

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How can we characterize the Galois closure of Σ "internally"? (meaning: how can you determine whether $\sigma \in \Sigma^{\perp\perp}$ without referring to structures?)

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where each α_i is an **axiom**, a member of Σ, or is derivable from earlier terms in the sequence using a **rule of inference**.

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2. (Generalization) $\frac{\varphi}{(\forall x_i)\varphi}$

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Thus $\Gamma \vdash \bot$. Now repeat the idea of Lindenbaum's Theorem with σ equal to $\neg((\exists x)\varphi(x) \to \varphi(c))$.]

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Applications of Compactness

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