

## The Compactness Theorem.

**Compactness Theorem.** *If  $\Sigma$  is a finitely satisfiable set of sentences, then  $\Sigma$  is satisfiable.*

The first stage of the proof is to enlarge  $\Sigma$  to a finitely satisfiable set of sentences having additional properties.

If  $\Lambda$  is a directed set, then a  $\Lambda$ -directed family is a  $\Lambda$ -indexed family of sets,  $(A_\lambda)_{\lambda \in \Lambda}$ , such that  $\mu < \nu$  implies  $A_\mu \subseteq A_\nu$ .

**Lemma 1.** *Let  $\Lambda$  be a directed set, let  $(\mathcal{L}_\lambda)_{\lambda \in \Lambda}$  be  $\Lambda$ -directed family of languages, and let  $(\Sigma_\lambda)_{\lambda \in \Lambda}$  be a  $\Lambda$ -directed family of sets of sentences with  $\Sigma_\lambda \subseteq \mathcal{L}_\lambda$ . If each  $\Sigma_\lambda$  is finitely satisfiable, then  $\bigcup_{\lambda \in \Lambda} \Sigma_\lambda$  is a finitely satisfiable set of  $\bigcup_{\lambda \in \Lambda} \mathcal{L}_\lambda$ -sentences.*

*Proof.* Any finite subset  $\Sigma_0$  of  $\bigcup_{\lambda \in \Lambda} \Sigma_\lambda$  lies in some  $\Sigma_\lambda$ , since  $\Lambda$  is directed.  $\Sigma_0$  can be satisfied by some  $\mathcal{L}_\lambda$ -structure,  $\mathbf{A}$ , since  $\Sigma_\lambda$  is finitely satisfiable and all symbols of  $\Sigma_\lambda$  belong to  $\mathcal{L}_\lambda$ . An arbitrary expansion of  $\mathbf{A}$  to the language  $\bigcup_{\lambda \in \Lambda} \mathcal{L}_\lambda$  will still satisfy  $\Sigma_0$ .  $\square$

**Lemma 2.** *If  $\Sigma$  is a finitely satisfiable set of  $\mathcal{L}$ -sentences, then there is a superset  $\Sigma' \supseteq \Sigma$  of  $\mathcal{L}$ -sentences that*

- (i) *is finitely satisfiable, and*
- (ii) *satisfies the **completeness condition** for  $\mathcal{L}$ , which is the statement that  $\sigma \in \Sigma'$  or  $(\neg\sigma) \in \Sigma'$  for every  $\mathcal{L}$ -sentence  $\sigma$ .*

*Proof.* Let  $\kappa = |\mathcal{L}|$  and let  $(\sigma_\lambda)_{\lambda < \kappa}$  be an enumeration of all  $\mathcal{L}$ -sentences. Define a sequence of sets of  $\mathcal{L}$ -sentences by

$$\begin{aligned} \Sigma_0 &:= \Sigma \\ \Sigma_{\lambda+1} &:= \begin{cases} \Sigma_\lambda \cup \{\sigma_\lambda\} & \text{if finitely satisfiable;} \\ \Sigma_\lambda \cup \{(\neg\sigma_\lambda)\} & \text{else.} \end{cases} \\ \Sigma_\lambda &:= \bigcup_{\mu < \lambda} \Sigma_\mu & \text{if } \lambda \text{ is limit.} \end{aligned}$$

$\Sigma_0$  is finitely satisfiable by assumption.

Suppose that  $\Sigma_\lambda$  is finitely satisfiable but  $\Sigma_{\lambda+1}$  is not. Then neither  $\Sigma_\lambda \cup \{\sigma\}$  nor  $\Sigma_\lambda \cup \{(\neg\sigma)\}$  is finitely satisfiable, meaning that there exists a finite (satisfiable) subset  $\Gamma \subseteq \Sigma$  such that neither  $\Gamma \cup \{\sigma\}$  nor  $\Gamma \cup \{(\neg\sigma)\}$  is satisfiable. If  $\mathbf{A}$  is a model of  $\Gamma$ , we are forced to conclude that neither  $\sigma$  nor  $(\neg\sigma)$  holds in  $\mathbf{A}$ , which is impossible.

Finally, if  $\Sigma_\mu$  is finitely satisfiable for all  $\mu < \lambda$ ,  $\lambda$  limit, then  $\Sigma_\lambda$  is finitely satisfiable by Lemma 1.

This proves that  $\Sigma' := \Sigma_\kappa$  is finitely satisfiable. Since each  $\mathcal{L}$ -sentence is some  $\sigma_\lambda$ , and  $\Sigma' (\supseteq \Sigma_{\lambda+1})$  contains either  $\sigma_\lambda$  or  $(\neg\sigma_\lambda)$ ,  $\Sigma'$  satisfies the completeness condition for  $\mathcal{L}$ .  $\square$

**Lemma 3.** *If  $\Sigma$  is a finitely satisfiable set of  $\mathcal{L}$ -sentences, then there are a set  $C$  of new constants, an extended language  $\mathcal{L}(C) = \mathcal{L} \cup C$ , and a superset  $\Sigma' \supseteq \Sigma$  consisting of  $\mathcal{L}(C)$ -sentences that*

- (i) *is finitely satisfiable, and*
- (ii) **has witnesses for  $\mathcal{L}$ -formulas**, which means that for every  $\mathcal{L}$ -formula  $\alpha(x)$  in one free variable there is a constant symbol  $c$  such that  $((\exists x\alpha(x)) \rightarrow \alpha(c)) \in \Sigma'$ .

*Proof.* Let  $\kappa = |\mathcal{L}|$  and let  $(\alpha_\lambda(x))_{\lambda < \kappa}$  be an enumeration of all  $\mathcal{L}$ -formulas with one free variable. Let  $C$  be a set of  $\kappa$ -many new constant symbols. Define a sequence of sets of  $\mathcal{L}(C)$ -sentences by

$$\begin{aligned}\Sigma_0 &:= \Sigma \\ \Sigma_{\lambda+1} &:= \Sigma_\lambda \cup \{((\exists x\alpha_\lambda(x)) \rightarrow \alpha(c_\lambda))\} \\ \Sigma_\lambda &:= \bigcup_{\mu < \lambda} \Sigma_\mu \quad \text{if } \lambda \text{ is limit.}\end{aligned}$$

Suppose that  $\Sigma_\lambda$  is finitely satisfiable but  $\Sigma_{\lambda+1}$  is not. Let  $\Gamma \subseteq \Sigma_\lambda$  be a finite set such that  $\Gamma \cup \{((\exists x\alpha_\lambda(x)) \rightarrow \alpha(c_\lambda))\}$  is unsatisfiable.  $\Gamma$  itself is satisfiable, so there must be a structure  $\mathbf{A}$  in an appropriate language that satisfies  $\Gamma$ ; we may assume that the language is  $\mathcal{L}(C_\lambda)$  where  $C_\lambda = \{c_\mu \mid \mu < \lambda\}$ . We claim that  $\mathbf{A}$  can be modified to an  $\mathcal{L}(C_{\lambda+1})$ -structure satisfying  $\Gamma \cup \{((\exists x\alpha_\lambda(x)) \rightarrow \alpha(c_\lambda))\}$ .

If  $\mathbf{A} \models (\exists x\alpha(x))$ ,  $\mathbf{A} \models \alpha[a]$  for some  $a \in A$ . Define  $c_\lambda^\mathbf{A} = a$ . If  $\mathbf{A} \not\models (\exists x\alpha(x))$ , then define  $c_\lambda^\mathbf{A} \in A$  arbitrarily. By interpreting  $c_\lambda$  in  $A$  we create a  $\mathcal{L}(C_{\lambda+1})$ -structure  $\mathbf{A}_c := \langle \mathbf{A}; c_\lambda \rangle$ . This structure still satisfies  $\Gamma$ , since  $c_\lambda$  does not appear in any sentence in  $\Gamma$ , but it now also satisfies  $((\exists x\alpha_\lambda(x)) \rightarrow \alpha(c_\lambda))$ . This contradicts our earlier supposition that  $\Gamma \cup \{((\exists x\alpha_\lambda(x)) \rightarrow \alpha(c_\lambda))\}$  is unsatisfiable.  $\square$

**Corollary 4.** *If  $\Sigma$  is a finitely satisfiable set of  $\mathcal{L}$ -sentences, then there are a set  $C$  of new constants, an extended language  $\mathcal{L}(C) = \mathcal{L} \cup C$ , and a superset  $\Sigma' \supseteq \Sigma$  consisting of  $\mathcal{L}(C)$ -sentences that*

- (i) *is finitely satisfiable, and*
- (ii) *satisfies the completeness condition for  $\mathcal{L}(C)$ , and*
- (iii) *has witnesses for  $\mathcal{L}(C)$ -formulas.*

*Proof.* Starting with  $\Sigma$ , we can use Lemmas 2 and 3 to create an increasing chain of finitely satisfiable sets  $\Sigma \subseteq \Sigma_c \subseteq \Sigma_{cw} \subseteq \Sigma_{cwc} \subseteq \Sigma_{cwcw} \subseteq \dots$  in increasing languages  $\mathcal{L} = \mathcal{L} \subseteq \mathcal{L}(C_w) = \mathcal{L}(C_w) \subseteq \mathcal{L}(C_{ww}) = \dots$ , where subscript  $c$  means “enforce the completeness condition” and subscript  $w$  means “add witnesses”. The union set  $\Sigma'$  will be finitely satisfiable and, with respect to the union language  $\mathcal{L}'$ , will satisfy the completeness condition and have witnesses.  $\square$

*Proof of the Compactness Theorem (really the second stage).* Call a set  $\Sigma$  of  $\mathcal{L}$ -sentences a **Henkin set** if it is finitely satisfiable, satisfies the completeness condition for  $\mathcal{L}$ , and has witnesses for  $\mathcal{L}$ -formulas. We will prove the Compactness Theorem in the special case where  $\Sigma$  is a Henkin set, then deduce that the theorem holds in general.

We proceed to construct a model of  $\Sigma$  under the assumption that it is a Henkin set.

**Step I.** (Defining the universe.)

Let  $C$  be the set of constant symbols of  $\mathcal{L}$ . Define a relation  $\sim$  on  $C$  by:  $c \sim d$  iff  $(c \doteq d) \in \Sigma$ .

**Claim 5.**  $\sim$  is an equivalence relation on  $C$ .

*Proof of claim.* If  $\sim$  were not reflexive on  $C$ , then there would be a  $c \in C$  such that  $(c \doteq c) \notin \Sigma$ . The completeness condition for  $\Sigma$  forces  $(\neg(c \doteq c)) \in \Sigma$ . By the finite satisfiability of  $\Sigma$  there must be an  $\mathcal{L}$ -structure  $\mathbf{X}$  satisfying  $(\neg(c \doteq c))$ . But this is impossible, since  $c$  interprets as an element  $c^{\mathbf{X}} \in X$ ,  $\doteq$  interprets as equality on  $X$ , and  $c^{\mathbf{X}} \neq c^{\mathbf{X}}$  never holds.

Similarly, if  $\sim$  failed to be symmetric or transitive on  $C$ , then applying the completeness condition and the finite satisfiability of  $\Sigma$  we obtain a contradiction.  $\square$

Write  $\bar{c}$  for  $c/\sim$  and  $\bar{C}$  for  $C/\sim$ . The universe of our model will be  $A := \bar{C}$ .

**Step II.** (Interpreting the  $\mathcal{L}$ -symbols.)

We will interpret the nonlogical symbols of  $\mathcal{L}$  according to the following rules.

(R) If  $R$  is an  $n$ -ary relation symbol, then define

$$(\bar{c}_0, \dots, \bar{c}_{n-1}) \in R^{\mathbf{A}} \quad \text{iff} \quad R(c_0, \dots, c_{n-1}) \in \Sigma.$$

(F) If  $F$  is an  $n$ -ary function symbol, then define

$$F^{\mathbf{A}}(\bar{c}_0, \dots, \bar{c}_{n-1}) = \bar{c}_n \quad \text{iff} \quad (F(c_0, \dots, c_{n-1}) \doteq c_n) \in \Sigma.$$

(c) If  $c$  is a constant symbol, then define  $c^{\mathbf{A}} = \bar{c}$ .

To eliminate any concerns about the validity of these definitions we prove the following

**Claim 6.** (1) For every term  $t$  with no variables there is an element  $c \in C$  such that  $(t \doteq c) \in \Sigma$ ; moreover,  $t^{\mathbf{A}} = c^{\mathbf{A}}$ .

(2) If for all subscripts  $i$  we have  $c_i \in C$ ,  $t_i$  is a term with no variables, and  $(t_i \doteq c_i) \in \Sigma$ , then  $R(t_0, \dots, t_{n-1}) \in \Sigma$  iff  $R(c_0, \dots, c_{n-1}) \in \Sigma$ .

(3) If for all subscripts  $i$  we have  $c_i \in C$ ,  $t_i$  is a term with no variables, and  $(t_i \doteq c_i) \in \Sigma$ , then  $(F(t_0, \dots, t_{n-1}) \doteq t_n) \in \Sigma$  iff  $(F(c_0, \dots, c_{n-1}) \doteq c_n) \in \Sigma$ .

*Proof of claim.* If  $t$  is a constant, the first part of Item (1) follows from the fact that  $\sim$  is reflexive. The second part follows from the fact that the equality relation on  $A$  is reflexive. Assume that  $t = F(t_0, \dots, t_{n-1})$  and that Item (1) is true for each  $t_i$ . Then there are constants  $c_i$  such that  $(t_i \doteq c_i) \in \Sigma$  and  $t_i^{\mathbf{A}} = c_i^{\mathbf{A}}$ . For the formula  $\alpha(x) := (F(c_0, \dots, c_{n-1}) \doteq x)$  there is a constant symbol  $c$  such that  $((\exists x \alpha(x)) \rightarrow \alpha(c)) \in \Sigma$ , since  $\Sigma$  has witnesses. The formula  $(\exists x \alpha(x))$  is valid, since  $F$  is an function symbol, so  $(\exists x \alpha(x)) \in \Sigma$ . It follows that  $\alpha(c) \in \Sigma$ , else by completeness  $\Sigma$  contains the unsatisfiable finite set

$$\{(\exists x \alpha(x)), ((\exists x \alpha(x)) \rightarrow \alpha(c)), (\neg \alpha(c))\}.$$

This shows that (i)  $(F(c_0, \dots, c_{n-1}) \doteq c) \in \Sigma$ , hence (ii)  $F^{\mathbf{A}}(c_0^{\mathbf{A}}, \dots, c_{n-1}^{\mathbf{A}}) = c^{\mathbf{A}}$ .

Regarding (i), the set

$$\{(t_0 \doteq c_0), \dots, (t_{n-1} \doteq c_{n-1}), (F(c_0, \dots, c_{n-1}) \doteq c), (\neg(F(t_0, \dots, t_{n-1}) \doteq c))\}$$

is unsatisfiable, and each sentence but the last belongs to  $\Sigma$ . By the completeness and finite satisfiability of  $\Sigma$  we have  $(F(t_0, \dots, t_{n-1}) \doteq c) \in \Sigma$  (or  $(t \doteq c) \in \Sigma$ ), as desired.

Regarding (ii), we have  $t^{\mathbf{A}} = F^{\mathbf{A}}(t_0^{\mathbf{A}}, \dots, t_{n-1}^{\mathbf{A}}) = F^{\mathbf{A}}(c_0^{\mathbf{A}}, \dots, c_{n-1}^{\mathbf{A}}) = c^{\mathbf{A}}$ . This completes the proof of Item (1).

If Item (2) were false, then by completeness  $\Sigma$  would contain either

$$\{(t_0 \doteq c_0), \dots, (t_{n-1} \doteq c_{n-1}), R(t_0, \dots, t_{n-1}), (\neg R(c_0, \dots, c_{n-1}))\}$$

or

$$\{(t_0 \doteq c_0), \dots, (t_{n-1} \doteq c_{n-1}), (\neg R(t_0, \dots, t_{n-1}))R(c_0, \dots, c_{n-1}), \}.$$

Both are unsatisfiable, so (2) is not false. A similar argument proves (3).  $\square$

**Step III.** (The  $\mathcal{L}$ -structure  $\mathbf{A} := \langle A; R, \dots, F, \dots, c, \dots \rangle$  satisfies  $\Sigma$ .)

**Claim 7.** *If  $\sigma$  is an  $\mathcal{L}$ -sentence, then  $\mathbf{A} \models \sigma$  iff  $\sigma \in \Sigma$ .*

*Proof of claim.* Assume first that  $\sigma$  is  $R(t_0, \dots, t_{n-1})$  for some  $R$ ,  $n$  and  $t_i$ . Since this is an atomic sentence, the  $t_i$  have no variables, hence Claim 6 (1) guarantees that there are constant symbols  $c_i$  such that  $(t_i \doteq c_i) \in \Sigma$  and  $t_i^{\mathbf{A}} = c_i^{\mathbf{A}}$ . We have the following equivalences:

$$\begin{aligned} \mathbf{A} \models \sigma & \text{ iff } \mathbf{A} \models R(t_0, \dots, t_{n-1}) \\ & \text{ iff } (t_0^{\mathbf{A}}, \dots, t_{n-1}^{\mathbf{A}}) \in R^{\mathbf{A}} \\ & \text{ iff } (c_0^{\mathbf{A}}, \dots, c_{n-1}^{\mathbf{A}}) \in R^{\mathbf{A}} \\ & \text{ iff } R(c_0, \dots, c_{n-1}) \in \Sigma \\ & \text{ iff } R(t_0, \dots, t_{n-1}) \in \Sigma \\ & \text{ iff } \sigma \in \Sigma. \end{aligned}$$

Each step follows from the definitions or from Claim 6.

The case where  $\sigma$  is  $(F(t_0, \dots, t_{n-1}) \doteq t_n)$  for some  $F$ ,  $n$  and  $t_i$  is similar.

Now assume that  $\sigma = (\alpha \wedge \beta)$ . We have the following equivalences:

$$\begin{aligned} \mathbf{A} \models (\alpha \wedge \beta) & \text{ iff } \mathbf{A} \models \alpha \text{ and } \mathbf{A} \models \beta \\ & \text{ iff } \alpha \in \Sigma \text{ and } \beta \in \Sigma \\ & \text{ iff } (\alpha \wedge \beta) \in \Sigma. \end{aligned}$$

The only step requiring justification is the equivalence of the last two lines. If we had  $\alpha, \beta \in \Sigma$  and  $(\alpha \wedge \beta) \notin \Sigma$ , then the completeness condition yields that  $\{\alpha, \beta, (\neg(\alpha \wedge \beta))\} \subseteq \Sigma$ , contrary to the finite satisfiability of  $\Sigma$ . In the reverse direction, if  $(\alpha \wedge \beta) \in \Sigma$  and, say,  $\alpha \notin \Sigma$ , then  $\{(\neg\alpha), (\alpha \wedge \beta)\} \subseteq \Sigma$  contradicts the finite satisfiability of  $\Sigma$ .

The case where  $\sigma = (\neg\alpha)$  can be argued similarly.

Finally we assume that  $\sigma = (\exists x \alpha(x))$ . We have the following equivalences:

$$\begin{aligned} \mathbf{A} \models (\exists x \alpha(x)) & \text{ iff } \mathbf{A} \models \alpha[a] \text{ for some } a \in A \\ & \text{ iff } \mathbf{A} \models \alpha(c) \text{ for some } c \in C \\ & \text{ iff } \alpha(c) \in \Sigma \text{ for some } c \in C \\ & \text{ iff } (\exists x \alpha(x)) \in \Sigma. \end{aligned}$$

We need to defend the claim that the last two lines are equivalent. The claim that  $\alpha(c) \in \Sigma$  for some  $c \in C$  implies  $(\exists x \alpha(x)) \in \Sigma$  follows from the completeness condition and the finite

satisfiability of  $\Sigma$ . The claim that  $(\exists x\alpha(x)) \in \Sigma$  implies  $\alpha(c) \in \Sigma$  for some  $c \in C$  follows from finite satisfiability and the fact that  $\Sigma$  has witnesses for  $\mathcal{L}$ -formulas.  $\square$

**Step IV.** (The theorem for general  $\Sigma$  follows from the theorem for  $\Sigma$  a Henkin set.)

Let  $\Sigma$  be a finitely satisfiable set of  $\mathcal{L}$ -sentences. According to Corollary 4 there is a language  $\mathcal{L}' \supseteq \mathcal{L}$  and a Henkin set  $\Sigma' \supseteq \Sigma$  in the language  $\mathcal{L}'$ . We have shown that  $\Sigma'$  has a model,  $\mathbf{A}$ . The reduct of  $\mathbf{A}$  to the symbols in  $\mathcal{L}$  is a model of  $\Sigma$ .  $\square$

**Corollary 8.** *An unsatisfiable set of sentences contains a finite unsatisfiable subset.*

*Proof.* This is the contrapositive of the Compactness Theorem.  $\square$

**Corollary 9.** *If  $\Sigma \models \sigma$ , then  $\Sigma_0 \models \sigma$  for some finite subset  $\Sigma_0 \subseteq \Sigma$ .*

*Proof.* Rewrite Corollary 8 using the equivalence of “ $\Sigma \models \sigma$ ” with “ $\Sigma \cup \{(\neg\sigma)\}$  is unsatisfiable” (which follows from the definition of  $\models$ ).  $\square$

**Corollary 10.** *If  $\Sigma$  has a model, then  $\Sigma$  has a model of size at most  $|\Sigma| + \omega$ .*

*Proof.* First, there is no harm in assuming that the only nonlogical symbols of  $\mathcal{L}$  are those that appear in  $\Sigma$ , so in particular  $|\mathcal{L}| = |\Sigma| + \omega$ . Next, show that the number of constant symbols added in the course of the proof is at most  $|\mathcal{L}|$ . Then  $|A| = |C| \sim |C| \leq |\mathcal{L}|$ .  $\square$

The case of Corollary 10 where  $\Sigma = \{\sigma\}$  (and hence  $|\Sigma| + \omega = \omega$ ) is called *Löwenheim’s Theorem*.

**Corollary 11.** *If  $\Sigma$  has arbitrarily large finite models, then  $\Sigma$  has an infinite model.*

*If  $\Sigma$  has an infinite model, then  $\Sigma$  has arbitrarily large infinite models.*

*Proof.* Expand the language to contain some infinite set  $C$  of new constant symbols, say  $\{c_\lambda \mid \lambda < \kappa\}$  where  $\kappa$  is an infinite cardinal. Let  $\Gamma = \{(\neg(c_\mu \doteq c_\nu)) \mid \mu < \nu < \kappa\}$ . If  $\Sigma$  has arbitrarily large finite models or an infinite model, then  $\Sigma \cup \Gamma$  is finitely satisfiable and is therefore satisfiable. Any model has cardinality at least  $\kappa$ .  $\square$

**Corollary 12.** *If  $\Sigma \cup \{\sigma\}$  is a set of  $\mathcal{L}$ -sentences and  $\sigma$  holds in every infinite model of  $\Sigma$ , then there is a finite  $N$  such that  $\sigma$  holds in every model of  $\Sigma$  of size greater than  $N$ .*

*Proof.* If this is not true, then  $\Sigma \cup \{(\neg\sigma)\}$  has arbitrarily large finite models and no infinite models, contrary to Corollary 11.  $\square$

**Corollary 13.** *The following conditions are equivalent for any class  $\mathcal{K}$  of  $\mathcal{L}$ -structures.*

- (1)  $\mathcal{K}$  is finitely axiomatizable.
- (2)  $\mathcal{K}$  is axiomatizable by a single  $\mathcal{L}$ -sentence.
- (3) Both  $\mathcal{K}$  and its complement are axiomatizable.

*Proof.* [(1) $\Rightarrow$ (2)] Replace a finite set of axioms for  $\mathcal{K}$  with its conjunction.

[(2) $\Rightarrow$ (3)] If  $\sigma$  axiomatizes  $\mathcal{K}$ , then  $(\neg\sigma)$  axiomatizes the complement of  $\mathcal{K}$ .

[(3) $\Rightarrow$ (1)] Suppose that  $\Sigma$  axiomatizes  $\mathcal{K}$  and  $\Gamma$  axiomatizes its complement. Then  $\Sigma \cup \Gamma$  is unsatisfiable, so there exist finite sets  $\Sigma_0 \subseteq \Sigma$  and  $\Gamma_0 \subseteq \Gamma$  such that  $\Sigma_0 \cup \Gamma_0$  is unsatisfiable.

I claim that  $\Sigma_0$  axiomatizes  $\mathcal{K}$ .  $\Sigma_0$  axiomatizes a class containing  $\mathcal{K}$ , since  $\Sigma_0 \subseteq \Sigma$ . But the containment cannot be proper, else there is some  $\mathbf{A} \notin \mathcal{K}$  such that  $\mathbf{A} \models \Sigma_0$ , and this would lead to  $\mathbf{A} \models \Sigma_0 \cup \Gamma_0$ , contrary to the conclusion of the last paragraph.  $\square$

**Corollary 14.** *If  $T_0 \subset T_1 \subset \dots$  is a strictly increasing chain of  $\mathcal{L}$ -theories, then the class of  $\mathcal{L}$ -structures  $\mathbb{A}$  such that  $\mathbb{A} \not\models T_i$  for some  $i$  is not elementary.*

*Proof.* The class identified in the corollary is the complement of the class axiomatized by  $\bigcup T_i$ , so it suffices to show that  $\bigcup T_i$  is not finitely axiomatizable. This follows from the fact that the chain  $T_0 \subset T_1 \subset \dots$  is strictly increasing and consists of theories.  $\square$

**Corollary 15.** *If  $\Sigma$  and  $\Gamma$  axiomatize the same class  $\mathcal{K}$  and  $\Gamma$  is finite, then there is a finite subset  $\Sigma_0 \subseteq \Sigma$  that axiomatizes  $\mathcal{K}$ .*

*Proof.* We may assume that  $\Gamma$  consists of a single sentence  $\gamma$ . Now  $(\neg\gamma)$  axiomatizes the complement of  $\mathcal{K}$ , so by the argument for Corollary 13 (3) $\Rightarrow$ (1) we get that some finite subset  $\Sigma_0 \subseteq \Sigma$  axiomatizes  $\mathcal{K}$ .  $\square$

**Corollary 16.** *If  $\mathbf{A}$  and  $\mathbf{B}$  are elementarily equivalent  $\mathcal{L}$ -structures, then there is a third  $\mathcal{L}$ -structure  $\mathbf{C}$  that is a common elementary extension of each of them.*

*Proof.* Let  $\mathbf{A}_A$  be the expansion of  $\mathbf{A}$  by constants and let  $\mathbf{B}_B$  be the expansion of  $\mathbf{B}$  by constants. Assume that all the new constant symbols are distinct. I claim that  $\Sigma := \text{Th}(\mathbf{A}_A) \cup \text{Th}(\mathbf{B}_B)$  is finitely satisfiable. To see this, note that any finite set of sentences from either theory is equivalent to a single one, so any finite set of sentences in  $\Sigma$  is equivalent to  $\alpha(\bar{a}) \wedge \beta(\bar{b})$  where (i)  $\alpha(\bar{x})$  and  $\beta(\bar{x})$  are  $\mathcal{L}$ -formulas, (ii) the coordinates of  $\bar{a}$  are symbols representing elements of  $A$  while the coordinates of  $\bar{b}$  are symbols representing elements of  $B$ , and (iii)  $\mathbf{A} \models \alpha(\bar{a})$  and  $\mathbf{B} \models \beta(\bar{b})$ .

Now  $\mathbf{B} \models \exists \bar{y} \beta(\bar{y})$ , so we also have  $\mathbf{A} \models \exists \bar{y} \beta(\bar{y})$ . Choose a tuple  $\bar{a}'$  such that  $\mathbf{A} \models \beta(\bar{a}')$ . Interpreting the symbols in  $\bar{b}$  in  $\mathbf{A}$  as  $\bar{a}'$  we get from  $\mathbf{A}$  a structure in the language  $\mathcal{L} \cup \{\bar{a}, \bar{b}\}$  satisfying  $\alpha(\bar{a}) \wedge \beta(\bar{b})$ , verifying the finite satisfiability of  $\Sigma$ .

Let  $\mathbf{C}$  be a model of  $\Sigma$ . Then  $\mathbf{C}$  is a model of  $\text{Th}(\mathbf{A}_A)$ , making  $\mathbf{C}$  an elementary extension of  $\mathbf{A}$ , similarly of  $\mathbf{B}$ .  $\square$

**Corollary 17.** *If  $\mathbf{A}$  is an  $\mathcal{L}$ -structure, then any set of types of  $\text{Th}(\mathbf{A})$  can be realized in some elementary extension of  $\mathbf{A}$ .*

**Applications.** (Terminology. The following are synonymous with each other: “finitely axiomatizable class”, “strictly elementary class”, “basic elementary class”, and “*EC*-class”. Any class that is the intersection of such classes is an “axiomatizable class”, an “elementary class”, or an “*EC* $_{\Delta}$ - (or *EC* $_{\delta}$ )-class”. The symbols  $\Delta/\delta$  refer to “Durchschnitt”.)

- (1) The class of finite sets (or finite  $\mathcal{L}$ -structures) is not elementary.
- (2) The class of infinite sets is elementary, but not finitely axiomatizable.
- (3) (Robinson's Principle) Let  $\Sigma \cup \{\sigma\}$  be a set of sentences in the language of fields. Assume that  $\Sigma$  contains the field axioms. Then  $\sigma$  is true in all models of  $\Gamma$  of characteristic zero iff  $\sigma$  is true in all models of  $\Gamma$  of characteristic  $p$  for sufficiently large  $p$ .
- (4) (Ax-Grothendieck Theorem) If  $V \subseteq \mathbb{C}^n$  is an algebraic variety, then any regular map  $\mathbf{m}: V \rightarrow V$  that is injective is also surjective.
- (5) The class of simple groups is not elementary.
- (6) The class of torsion groups is not elementary.
- (7) If  $G$  is a group with elements of arbitrarily large finite order, then  $G$  is elementarily equivalent to a group  $H$  that has an element of infinite order.
- (8) There is no formula  $\tau(x)$  in the language of groups that defines the set of torsion elements of a group. (I.e., no  $\tau(x)$  such that, for every group  $G$ ,  $G \models \tau[g]$  iff  $g$  is a torsion element.)
- (9) The class of torsion free groups is elementary, but not finitely axiomatizable.
- (10) The class of algebraically closed fields of characteristic zero is axiomatizable, but not finitely axiomatizable.
- (11) ( $\exists$  nonstandard models of  $\text{Th}(\mathbb{R})$ ) The field of real numbers is elementarily equivalent to an ordered field  $\mathbb{F}$  that contains a positive infinitesimal element.
- (12) The property of  $\mathbb{R}$  that every nonempty subset with an upper bound has a least upper bound cannot be expressed by any set of first-order sentences.
- (13) ( $\exists$  nonstandard models of  $\text{Th}(\mathbb{N})$ ) The structure  $\langle \mathbb{N}; \cdot, + \rangle$  is elementarily equivalent to a structure with infinitely large elements (in fact, infinitely large prime numbers).
- (14) There are  $2^{\aleph_0}$  nonisomorphic countable structures elementarily equivalent to  $\langle \mathbb{N}; \cdot, + \rangle$ .
- (15) The class of well-ordered sets is not elementary.
- (16) Every partial order can be extended to a linear order.
- (17) The class of connected graphs is not elementary.
- (18) There is no first-order formula  $\gamma(x, y)$  in the language of graphs that defines the relation "there is a path from  $x$  to  $y$ ".
- (19) The class of acyclic graphs is elementary, but not finitely axiomatizable.
- (20) (Erdős) A graph is  $k$ -colorable iff each finite subgraph is  $k$ -colorable.
- (21) (König's Lemma) Any infinite, finitely branching tree has an infinite path.