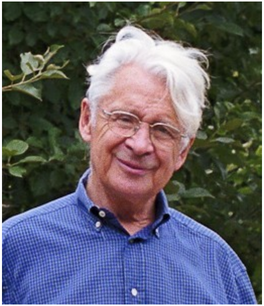
ω-categorical structures



Countable categoricity

Throughout these slides, T will be a complete theory in a countable language which has infinite models.

We are interested in the question: what can be said about T (and its models) if T is ω-categorical? (Synonym: $\aleph_0$-categorical.)

Recall that a theory is κ-categorical if it has one isomorphism type of model of size κ. A structure is κ-categorical if its theory is.

Immediate observations. (Assume T is ω-categorical.)

1. T has countably many (= 1) isomorphism types of countable models, so T is "small". ($|S_n(T)| \leq \aleph_0$ for each n.)
2. T has a countable atomic model and a countable ω-saturated model, which must be isomorphic.
3. Every type in $S_n(T)$ is isolated. Hence $S_n(T)$ is finite for all n. (Discreteness+compactness)

The theorems of Erwin Engeler, Czesław Ryll-Nardzewski, and Lars Svenonius, 1959

Theorem. Let T be a complete theory in a countable language which has infinite models. TFAE

1. T is ω-categorical.
2. $S_n(T)$ is finite for every n.
3. Every (or some) countable model of T is both atomic and ω-saturated.
4. Every (or some) countable model of T has an "oligomorphic" automorphism group.

The first three conditions are equivalent. From the previous slide, we have

- $(1) \Rightarrow (3)$(every).
- (3)(some) $\Rightarrow (2)$.
- $(2) \Rightarrow$ all countable models are atomic $\Rightarrow (1)$.

Oligomorphic group actions

Oligo- is a Greek prefix meaning "few".

If a group G acts on a set X, then it acts "diagonally" on X^n for any n. Diagonal action means, if $g \in G$ and $(x_1, \ldots, x_n) \in X^n$, then

$$g \cdot (x_1, \ldots, x_n) \quad \text{is defined to be} \quad (g \cdot x_1, \ldots, g \cdot x_n).$$

The action of G on X is **oligomorphic** if the number of G-orbits of X^n is finite for every n. ("Few" orbits.)

Simplest example. Let X be an infinite set and let $G = \mathrm{Sym}(X)$ be the group of all permutations of the set X. G acts oligomorphically on X. (Describe the orbits of G on X, X^2, X^3, ETC.) What happens if you replace $\mathrm{Sym}(X)$ by its subgroup $\mathrm{Sym}_{\mathrm{fin}}(X)$ of permutations of finite support?

Next simplest example. Let $G = \mathrm{Aut}(V)$ where V is an ω-dimensional vector space over a finite field $\mathbb{F}$. G acts oligomorphically on V.

Svenonius's "automorphism version" of the theorem

Theorem. Let T be a complete theory in a countable language which has infinite models. TFAE

(2) $S_n(T)$ is finite for every n.

(4) Every (or some) countable model of T has an oligomorphic automorphism group.

Proof.

- $(2) \Rightarrow (4)$(every): Assume that $\mathbf{M}$ is any countable model of T. $\mathbf{M}$ must be atomic. By strong ω-homogeneity, any two tuples in $\mathbf{M}^n$ have the same type iff they belong to the same orbit of $\mathrm{Aut}(\mathbf{M})$. Since $S_n(T)$ is finite, the action of $\mathrm{Aut}(\mathbf{M})$ is oligomorphic.
- (4)(some) $\Rightarrow (2)$: Let $\mathbf{M}$ be a countable model of T that has an oligomorphic automorphism group. $\mathbf{M}$ can realize only finitely many n-types for any n, yet $\mathbf{M}$ realizes a dense set of n-types, so $S_n(T)$ is finite for every n. $\square$

Other characterizations of ω-categoricity

Let T be a complete theory in a countable language which has infinite models. Each of the following statements about T is equivalent to the statement that T is ω-categorical.

1. T has a countable model that is both atomic and saturated.
2. Every model of T realizes only isolated types.
3. For each natural number n, there are only finitely many n-variable formulas up to T-equivalence. Here we say that $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are T-equivalent (or equivalent modulo T) if

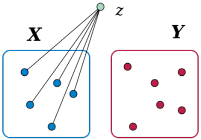
$$T \models (\forall \mathbf{x})(\alpha(\mathbf{x}) \leftrightarrow \beta(\mathbf{x}))$$

4. T has a model $\mathbf{M}$ such that, for every n, M^n has finitely many 0-definable subsets.

The random graph is an ω-categorical structure

Example. Let's recall the definition of the random graph.
This is a graph with vertex set $V = \omega$. Decide whether the graph to be constructed has an edge between $i, j \in V$ by flipping a coin. The graph so constructed will "almost surely" (= "with probability 1") satisfy the following first-order properties:

P(m,n). Whenever $X, Y \subseteq V$ are disjoint subsets with $|X| = m$ and $|Y| = n$, there is a $z \in V - (X \cup Y)$ that is adjacent to every vertex in X and not adjacent to any vertex in Y.



A "q.f. type extension" property

The theory axiomatized by $\{P(m, n) \mid m, n \in \omega\}$ is complete and ω-categorical. Its unique countable model is called the **random graph**.

Comments

1. The "q.f. type extension" axioms guarantee that the random graph is ultrahomogeneous.
2. Hence the theory has q.e.
3. Hence the elementary type of a tuple $\mathbf{v}$ is determined by the isomorphism type of the subgraph induced on the coordinate values of $\mathbf{v}$ and the duplications in coordinate values. This is enough to show that $S_n(T)$ is finite for every n. This explains why the theory of the random graph is ω-categorical.
4. These arguments generalize to show that the theory of any countably infinite ultrahomogeneous structure in a finite relational language has q.e. and is ω-categorical.
5. For ω-categorical structures, q.e. is equivalent to ultrahomogeneity.