Atomic models

Recall

We have been trying to determine the extent to which a structure is determined by its language.

So far we have learned:

(1) A finite structure is determined up to isomorphism by its complete first-order theory. (HW)

(2) An infinite structure can never be determined up to isomorphism by its complete first-order theory. (Löwenheim-Skolem Theorem)

(3) Some infinite structures may be determined up to isomorphism by its cardinality and its theory. (Concept of κ-categoricity)

We are in the process of investigating how:

(4) The variety of isomorphism types of models of a complete theory T is determined by the spaces $S_n(T)$.

Models realizing few types

Throughout these slides, T will be a complete theory in a countable language which has infinite models.

Definition. Call a model of T *atomic* if it realizes only isolated types.

Theorem of Atomic Models. Let T be a complete theory in a countable language. TFAE.

1. T has a countable atomic model,
2. The isolated types in $S_n(T)$ are dense for all $n \in \omega$.

Remarks.

1. Hence, it is common to define a complete theory in a countable language to be *atomic* if its isolated types are dense.
2. Countable atomic models are unique up to isomorphism, and are elementarily embeddable into every other model.
3. Henkin models are atomic.
4. Lots of other theories have atomic models. (Any ω-categorical theory, Any T satisfying $|S_n(T)| < 2^{\aleph_0}$ for all n, $T = \mathrm{Th}(\langle \omega; +, \cdot \rangle)$.)

Proof of the Theorem of Atomic Models

Statement.
T has a (countable) atomic model $\Leftrightarrow$ isolated types are dense in $S_n(T)$, $\forall n$.

($\Rightarrow$:) Assume T has an atomic model, $\mathbf{A}$. $\mathbf{A}$ realizes only isolated types in $S_n(T)$, because $\mathbf{A}$ is atomic. But $\mathbf{A}$ realizes a dense subset of $S_n(T)$, since T is complete. Hence isolated types are dense in $S_n(T)$.

($\Leftarrow$:) Now assume that isolated types, e.g. $p = \langle \varphi_p(\mathbf{x}) \rangle$, are dense in $S_n(T)$ for every n. For a given n, define (in order to omit!)

$$\Sigma_n(\mathbf{x}) = \{\neg\varphi_p(\mathbf{x}) \mid p \in S_n(T) \text{ isolated by } \varphi_p\}.$$

If $\Sigma(\mathbf{x})$ is not consistent with T, then it is <u>omitted</u> in all models of T. ☺
If $\Sigma(\mathbf{x})$ is consistent and supported by $\psi(\mathbf{x})$, then $O_{\psi(\mathbf{x})}$ would be nonempty, clopen, but contain no isolated type, contradicting density. ☹
Else $\Sigma_n(\mathbf{x})$ is an unsupported partial type for each n. There are countably many Σ_n. By the Omitting Types Theorem, T has a countable model $\mathbf{A}$ omitting all Σ_n. $\mathbf{A}$ is countable and atomic. □

A non-atomic theory

Example.
The theory T of countably many independent unary relations has these properties:

1. Axiomatized by all sentences

$$\exists x(R_{i_1}(x) \wedge \cdots \wedge R_{i_m}(x) \wedge \neg R_{j_1}(x) \wedge \cdots \wedge \neg R_{j_n}(x)).$$

2. T is complete and has q.e.
3. $S_n(T)$ has no isolated points for any n.

The word "atomic"

A Boolean algebra is called *atomic* if every nonzero element is above an atom.

The Boolean algebra of clopen subsets of a Stone space is therefore atomic if every clopen subset contains a singleton clopen subset. This is equivalent to the property that isolated points are dense. Hence the theory T is atomic iff the Boolean algebra associated to each $S_n(T)$ is atomic.

There is a nice sufficient condition for a Stone space to have isolated points dense.

Thm. If S is a scattered Stone space, then its isolated points are dense.

Proof.(Contrapositive) Assume that C is a nonempty basic clopen subset of S not containing any point isolated in S. Since C is open, it has no points isolated in C. Since C is closed without isolated points, it is a nonempty perfect subset of S. $\square$

Uniqueness of atomic models

Thm. If T is a complete atomic theory in a countable language, then

1. Any two countable atomic models are isomorphic.
2. Any two tuples of the atomic model have the same type iff they differ by an automorphism.
3. The unique countable model is “prime” (is elementarily embeddable in any model of T).

Proof. (1) Back and forth. (2) Back and forth. (3) Forth. □

An extension lemma

Lm. Let T be a complete theory. Let $p \in S_n(T)$ and $q \in S_{n+1}(T)$ be types where $p = q|_n$ is the restriction of q to the first n variables. If p and q are both isolated types, then

> *for any model* $\mathbf{M}$ *of* T*, any realization* $\mathbf{a}$ *of* p *in* $\mathbf{M}$ *can be extended to a realization* $\mathbf{a}b$ *of* q *in* $\mathbf{M}$*.*

Proof. Assume that p and q are isolated by $\varphi_p(\mathbf{x})$ and $\varphi_q(\mathbf{x}, x_{n+1})$. Then $(\exists x_{n+1})\varphi_q(\mathbf{x}, x_{n+1}) \in q|_n = p$, so

$$T \models (\forall \mathbf{x})(\varphi_p(\mathbf{x}) \to (\exists x_{n+1})\varphi_q(\mathbf{x}, x_{n+1})).$$

Hence any realization of p in any model of T can be extended to a realization of q. $\square$