Atomic models

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We are in the process of investigating how:

(4) The variety of isomorphism types of models of a complete theory T is determined by the spaces $S_n(T)$.

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