

March 13, Key points. The structure of algebraically closed fields.

Every field can be constructed through a series of extensions of its prime subfield.

- (1) A **field** is a structure $\mathbb{F} = \langle F; +, -, 0, \cdot, 1 \rangle$ satisfying the axioms of fields. (The field axioms assert that $\langle F; +, -, 0 \rangle$ is an abelian group, $\langle F; \cdot, 1 \rangle$ is a monoid, multiplication distributes over addition, and every nonzero element of \mathbb{F} has a multiplicative inverse.)
- (2) The **prime subfield** of \mathbb{F} is the least subfield of \mathbb{F} , equivalently it is the intersection of all subfields, equivalently it is the subfield generated by $\{0, 1\}$. The prime subfield of any field is isomorphic to \mathbb{Q} or to the unique field of p elements for some prime p .

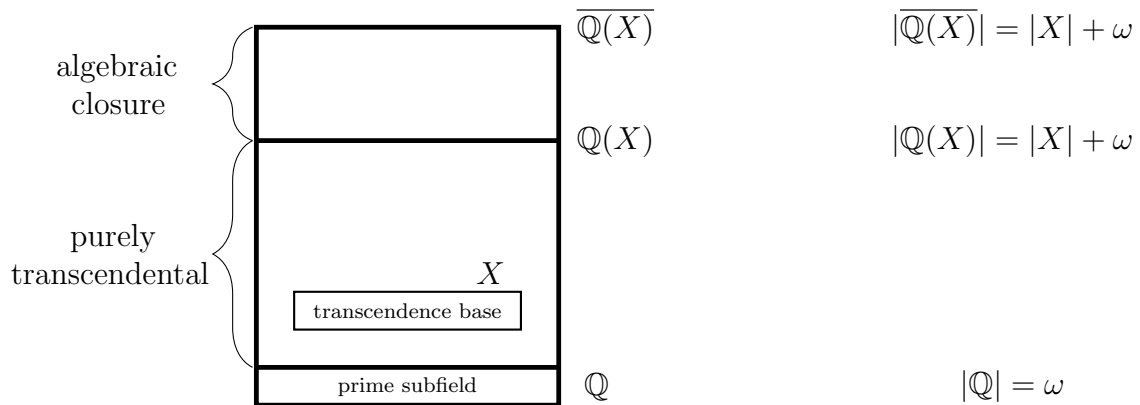
There are first-order sentences in the language of fields which assert that a structure is an algebraically closed field of characteristic zero. They are:

- (1) the sentences defining fields, indicated above,
- (2) the sentences expressing “characteristic is zero”, which are $\underbrace{“1 + \cdots + 1 \neq 0”}_{k \text{ terms}}$, one such sentence for every $k > 0$,
- (3) the sentences expressing that a field is “algebraically closed”, which are

$$(\forall y_0) \cdots (\forall y_{n-1})(\exists x)(x^n + y_{n-1}x^{n-1} + \cdots + y_1x + y_0 = 0),$$

one such sentence for every $n > 0$.

THE STRUCTURE AND SIZE OF AN ALGEBRAICALLY CLOSED FIELD WITH $\text{CHAR}(\mathbb{F}) = 0$



Remarks.

- (1) Let \mathbb{F} a field of characteristic zero, and let $X \subseteq \mathbb{F}$ be a maximal algebraically independent subset. (This exists by Zorn's Lemma.) The cardinality $|X|$ is uniquely determined, and it is called the transcendence degree of \mathbb{F} over \mathbb{Q} .

- (2) Let $\overline{\mathbb{F}}$ be an algebraic closure of \mathbb{F} . Replacing fields by isomorphic copies if necessary, we may assume that

$$\mathbb{Q}(X) \subseteq \mathbb{F} \subseteq \overline{\mathbb{F}} = \overline{\mathbb{Q}(X)}.$$

Thus, whatever field of characteristic zero we are working in, we may assume that we are working in a part of $\overline{\mathbb{Q}(X)}$.

- (3) Since algebraically closed fields have the form $\overline{\mathbb{Q}(X)}$, and since $|X|$ is uniquely determined by the field, it follows that there is one isomorphism type of algebraically closed field of characteristic zero for every cardinal $|X|$.
- (4) Altogether, we have countably many algebraically closed fields of characteristic zero that have countable cardinality: $\overline{\mathbb{Q}(X)}$ for $|X| = 0, 1, 2, \dots, \omega$. Then we have one of cardinality κ for every infinite κ : $\overline{\mathbb{Q}(X)}$ for $|X| = \kappa$.
- (5) All of these statements remain true if “characteristic zero” is replaced by “characteristic p ”, except the prime field of characteristic zero, \mathbb{Q} , should be replaced by the prime field of characteristic p , \mathbb{F}_p .
- (6) Let $\overline{\mathbb{F}}_p$ be the algebraic closure of the p -element field. An ultraproduct $\prod_{\mathcal{U}} \overline{\mathbb{F}}_p$ over a nonprincipal ultrafilter on ω , with one factor of each type $\overline{\mathbb{F}}_p$, will be an algebraically closed field of characteristic zero of size 2^{\aleph_0} . Necessarily it will be isomorphic to the unique algebraically closed field of characteristic zero of size 2^{\aleph_0} , which is \mathbb{C} .