The structure of algebraically closed fields. Key points.

Every field can be constructed through a series of extensions of its prime subfield.

(1) A **field** is a structure $\mathbb{F} = \langle F; +, -, 0, \cdot, 1\rangle$ satisfying the axioms of fields. (The field axioms assert that $\langle F; +, -, 0\rangle$ is an abelian group, $\langle F; \cdot, 1\rangle$ is a monoid, multiplication distributes over addition, and every nonzero element of $\mathbb{F}$ has a multiplicative inverse.)

(2) The **prime subfield** of $\mathbb{F}$ is the least subfield of $\mathbb{F}$, equivalently it is the intersection of all subfields, equivalently it is the subfield generated by $\{0, 1\}$. The prime subfield of any field is isomorphic to $\mathbb{Q}$ or to the unique field of p elements for some prime p.

There are first-order sentences in the language of fields which assert that a structure is an algebraically closed field of characteristic zero. They are:

(1) the sentences defining fields, indicated above,

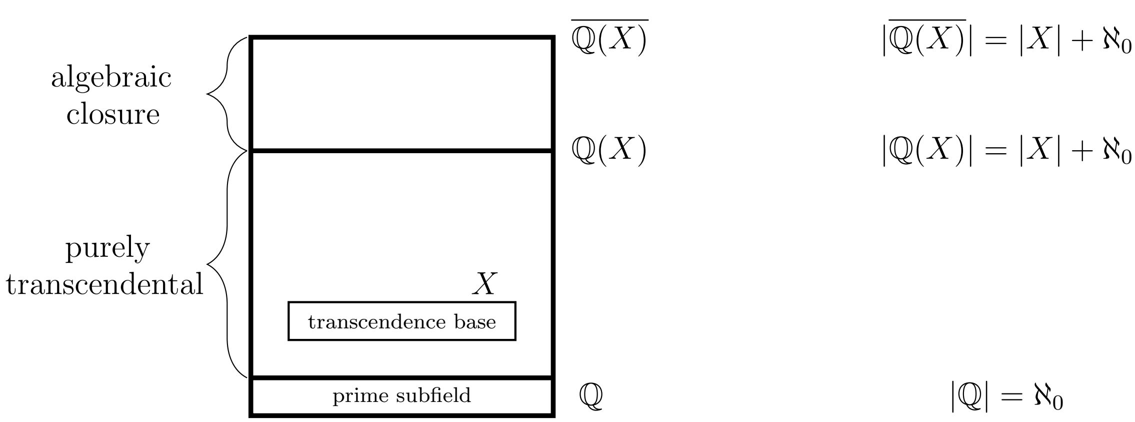
(2) the sentences expressing "characteristic is zero", which are "$\underbrace{1 + \cdots + 1}_{k \text{ terms}} \neq 0$", one such sentence for every $k > 0$,

(3) the sentences expressing that a field is "algebraically closed", which are

$$(\forall y_0)\cdots(\forall y_{n-1})(\exists x)(x^n + y_{n-1}x^{n-1} + \cdots + y_1 x + y_0 = 0),$$

one such sentence for every $n > 0$.

The structure and size of an algebraically closed field with char($\mathbb{F}$) = 0



Remarks. Let $\mathbb{F}$ a field of characteristic zero.

(1) Let $X \subseteq \mathbb{F}$ be a maximal algebraically independent subset. (This exists by Zorn's Lemma.) The cardinality $|X|$ is uniquely determined, and it is called the transcendence degree of $\mathbb{F}$ over $\mathbb{Q}$.

(2) Let $\overline{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$. Replacing fields by isomorphic copies if necessary, we may assume that

$$\mathbb{Q}(X) \subseteq \mathbb{F} \subseteq \overline{\mathbb{F}} = \overline{\mathbb{Q}(X)}.$$

Thus, whatever field of characteristic zero we are working in, we may assume that we are working in a part of $\overline{\mathbb{Q}(X)}$.

(3) Since algebraically closed fields have the form $\overline{\mathbb{Q}(X)}$, and since $|X|$ is uniquely determined by the field, it follows that there is one isomorphism type of algebraically closed field of characteristic zero for every cardinal $\kappa > \aleph_0$.

(4) Altogether, we have countably many algebraically closed fields of characterstic zero that have countable cardinality: $\overline{\mathbb{Q}(X)}$ for $|X| = 0, 1, 2, \ldots, \omega$. Then we have one of cardinality κ for every infinite κ: $\overline{\mathbb{Q}(X)}$ for $|X| = \kappa$.

(5) All of these statements remain true if "characteristic zero" is replaced by "characteristic p", except the prime field of characteristic zero, $\mathbb{Q}$, should be replaced by the prime field of characteristic p, $\mathbb{F}_p$.

(6) Let $\overline{\mathbb{F}}_p$ be the algebraic closure of the p-element field. An ultraproduct $\prod_{\mathcal{U}} \overline{\mathbb{F}}_p$ over a nonprincipal ultrafilter on ω, with one factor of each type $\overline{\mathbb{F}}_p$, will be an algebraically closed field of characteristic zero of size $2^{\aleph_0}$. Necessarily it will be isomorphic to the unique algebraically closed field of characteristic zero of size $2^{\aleph_0}$, which is $\mathbb{C}$.