MODEL THEORY
HOMEWORK ASSIGNMENT III

Read Chapter 3.

Problems 1-5 are due April 12. Problems 6-10 are due April 19.

PROBLEMS

1. (Ari)
Prove that if L-structures $\mathbb{A}$ and $\mathbb{B}$ are elementarily equivalent, then there exists an L-structure $\mathbb{C}$ into which both can be elementarily embedded.
[Hint: There is no harm in assuming that $A \cap B = \emptyset$. Then show that $\mathbf{Th}(\mathbb{A}_A) \cup \mathbf{Th}(\mathbb{B}_B)$ is a consistent $L(A \cup B)$-theory.]

2. (Oscar)
Suppose that L is a language and L' is an expansion of L by some set C of additional constant symbols. Suppose that T is an L-theory that has quantifier elimination and that $T' \supseteq T$ is an L'-theory extending T. Show that T' has q.e.
The theory of dense linear order without endpoints has q.e. (you may assume this). Show that any theory of dense linear order with some additional constant symbols is complete iff the theory completely decides how the order relation restricts to the interpretations of the constant symbols.

3. (Silas)
A structure is *ultrahomogeneous* if every isomorphism between finitely generated substructures extends to an automorphism. Show that if $\mathbb{A}$ is a finite L-structure, then $\mathbf{Th}(\mathbb{A})$ has quantifier elimination iff $\mathbb{A}$ is ultrahomogeneous.

4. (Khizar)
The theory T of countably many independent unary relations is the theory in the language with relation symbols $R_n(x)$, $n < \omega$, which contains all sentences of the form

$$\exists x(R_{i_1}(x) \wedge \cdots \wedge R_{i_m}(x) \wedge \neg R_{j_1}(x) \wedge \cdots \wedge \neg R_{j_n}(x))$$

whenever $i_1, \dots, i_m, j_1, \dots, j_n$ are distinct.

Show that T has quantifier elimination and is complete.

5. (Trevor)

Let L be the language whose only nonlogical symbol is one unary relation symbol. Find all L-theories that have quantifier elimination. Are they all complete?

6. (Ari, Trevor)

Which finite abelian groups A have the property that Th(A) has quantifier elimination?
[Hint: Make some use of Problem 4. Prove that a finite abelian group is ultrahomogeneous if and only if its Sylow subgroups are ultrahomogeneous and then determine which finite, abelian, p-groups are ultrahomogeneous.]

7. (Oscar, Khizar)

Which finite, simple, unital rings R have the property that Th(R) has quantifier elimination?
[Hints: Use the Wedderburn-Artin Theorem and Wedderburn's Little Theorem to show that $R \cong M_n(\mathbb{F}_q)$ for some positive integer n and some prime power q. To complete the problem, you must find the values of n and q associated to a ring whose theory has q.e. It may help to know that every automorphism of R is inner (see the Skolem-Noether Theorem).

One thing you will have to prove as you solve this exercise is that every finite field is ultrahomogeneous. For this, you will need to know something about the subfield structure and the automorphism group of a finite field.]

8. (Silas, Khizar)

Let T be the theory of $\mathbb{A} = \langle \omega; \cdot, + \rangle$, and show that $|S_1(T)| = 2^\omega$. Conclude that there are 2^ω countable models of T up to isomorphism.

9. (Oscar, Trevor)

Let $\mathbb{Q}$ be the field of rational numbers. Show that Th$(\mathbb{Q})$ has continuumly many countable models up to isomorphism.

10. (Ari, Silas)

Prove that, if ZFC is consistent, then it has continuumly many countable models up to isomorphism.