# MODEL THEORY
# HOMEWORK ASSIGNMENT II

Read Chapter 2.

## PROBLEMS

**1. (Ari, Khizar) Let $\mathrm{Th}_L$ be the lattice of all $L$-theories for some language $L$. Show that any atom in this lattice has a complement, any complement of an atom is a coatom (and vice versa), but that there must exist at least one coatom that does not have a complement.**

**2. (Oscar, Pras) Let $L$ be a language and let $X = \mathrm{Spec}(L)$ be its space of complete theories. Show that for any ordinal $\alpha$ there is a theory $T_\alpha$ such that $V(T_\alpha)$ is the closed set of all complete $L$-theories of Cantor-Bendixson rank at least $\alpha$. Do this by describing how to generate a set of axioms for $T_\alpha$.**

**3. (Silas, Trevor) Let $n > 0$ be chosen and fixed. Let $L$ be the language of one binary predicate, $E(x, y)$, and let $T$ be the $L$-theory of one equivalence relation with $n$ classes. That is, $T$ is axiomatized by the sentences asserting that $E$ defines a reflexive, symmetric, transitive binary relation, along with a sentence**

$$(\exists x_1)\cdots(\exists x_n)\left(\left(\bigwedge_{i\neq j}\neg E(x_i, x_j)\right)\wedge(\forall y)\left(\bigvee_i E(x_i, y)\right)\right)$$

**which asserts that $E$ has exactly $n$ classes.**

**The purpose of this problem is to investigate the Cantor-Bendixson rank of the closed subset $V(T) \subseteq \mathrm{Spec}(L)$ consisting of the complete theories that extend $T$.**

**(a) Describe the complete theories of Cantor-Bendixson rank $0$ in $V(T)$.**
**(b) Describe the complete theories of Cantor-Bendixson rank $1$ in $V(T)$.**
**(c) Make a conjecture about the Cantor-Bendixson rank of $V(T)$.**

**4. (Ari, Khizar) Show that the number of ultrafilters on an infinite set $I$ is $2^{2^{|I|}}$, by proving the following two statements.**

**(a) # ultrafilters $\leq 2^{2^{|I|}}$.**
**(b) # ultrafilters $\geq 2^{2^{|I|}}$.**

**Hints for (b): Let $\mathcal{F}$ be the set of finite subsets of $I$ and let $\Phi$ be the set of finite subsets of $\mathcal{F}$. For each subset $J \subseteq I$ define**

$$A_J = \{(f, \phi) \in \mathcal{F} \times \Phi \mid J \cap f \in \phi\}.$$

**Let $A_j^c := (\mathcal{F} \times \Phi) - A_j$ be the complement of $A_J$. For each subset $S \subseteq \mathcal{P}(I)$, show that the set $\mathcal{A}_S = \{A_J \mid J \in S\} \cup \{A_J^c \mid J \notin S\}$ has the Finite Intersection Property. Thus each $\mathcal{A}_S$ extends to some ultrafilter $\mathcal{U}_S$. On the other hand, prove that $R \neq S$ implies $\mathcal{U}_R \neq \mathcal{U}_S$ by arguing that if $R, S \subseteq \mathcal{P}(I)$ and $J \in R - S$, then $A_J \in \mathcal{U}_R - \mathcal{U}_S$.**

**5. (Oscar, Pras) An (nonprincipal) ultrafilter is *uniform* if all of its sets have the same size. Show that a regular ultrafilter is uniform.**

**6. (Silas, Trevor) Let $G$ be the infinite (connected) graph whose vertex set is $\mathbb{Z}$ and whose adjacency relation relates each integer only to its immediate successor and its immediate predecessor. Let $2G$ denote the graph that is equal to two disjoint copies of $G$. Show that $G$ and $2G$ have isomorphic ultrapowers. Deduce that "connectivity" is not a first-order expressible property of graphs.**

**7. (Ari, Khizar) Consider the set $A = \{\mathbb{Z}_n \mid n \in \mathbb{N}^+\}$, where $\mathbb{Z}_n$ denotes the cyclic group of order $n$. Let $\mathcal{U}$ be an ultrafilter on a set $\mathbb{N}^+$, and let $G = \prod_{\mathcal{U}} \mathbb{Z}_n$ be the ultraproduct of the family $A$ with respect to $\mathcal{U}$. Show that the torsion subgroup of $G$ is isomorphic to a subgroup of $\mathbb{Q}/\mathbb{Z}$. Explain how to choose $\mathcal{U}$ so that the torsion subgroup of $G$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$.**

**8. (Oscar, Pras) Let $\mathcal{K}$ be a class of $L$-structures. Show that an ultraproduct of ultraproducts of members of $\mathcal{K}$ is isomorphic to an ultraproduct of members of $\mathcal{K}$.**

**9. (Silas, Trevor) Let $L$ be a language in a signature $(\mathcal{C}, \mathcal{F}, \mathcal{R}, \mathrm{ar})$. A class of $L$-structures is pseudo-elementary if it is elementary in some richer signature. ("Richer" means "richer or equal".)**

- **(a) Show that, in the language of one unary predicate $P(x)$, the class of structures $\langle A; P(x)\rangle$ where $|P[A]| = |A - P[A]|$ is pseudo-elementary but not elementary.**
- **(b) Show that pseudo-elementary classes are closed under ultraproducts.**

**10. (Ari, Khizar)**

- **(a) Let $\mathcal{U}$ be an ultrafilter on a set $I$, and let $\{\mathbb{A}_i \mid i \in I\}$ be a set of $L$-structures. Show that $\prod_{\mathcal{U}} \mathrm{Aut}(\mathbb{A}_i)$ is embeddable in $\mathrm{Aut}(\prod_{\mathcal{U}} \mathbb{A}_i)$. (Elements of $\prod_{\mathcal{U}} \mathrm{Aut}(\mathbb{A}_i)$ are called internal automorphisms of $\prod_{\mathcal{U}} \mathbb{A}_i$, while other automorphisms are called external.)**

**(b) Give an explicit example of an external automorphism.**

**11. (Oscar, Pras) Show that the following conditions on $\mathbb{A}$ are equivalent.**

**(a) $\mathbb{A}$ is a model of the theory of the class of finite $L$-structures.**
**(b) Every $L$-sentence true in $\mathbb{A}$ holds in some finite $L$-structure.**
**(c) $\mathbb{A}$ is elementarily equivalent to an ultraproduct of finite $L$-structures.**

**($\mathbb{A}$ is *pseudofinite* if these hold.)**

**12. (Silas, Trevor) Show that any structure is embeddable in an ultraproduct of its finitely generated substructures. Conclude that any universal class is generated by its finitely generated members. Show that this statement about universal classes is not true for arbitrary elementary classes. (Universal class = a class axiomatizable by universally quantified sentences.)**